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INDUCTIVE DEFINITIONS AND REFLECTING PROPERTIES OF ADMISSIBLE ORDINALS

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Introduction.

An operator or inductive definition (i.d.) $\Gamma: P(\omega) \to P(\omega)$ determines a transfinite sequence $\langle \Gamma^\xi : \xi \in ON \rangle$ of subsets of ω, where $\Gamma^\lambda = \cup\{\Gamma(\Gamma^\xi) : \xi < \lambda\}$. The closure ordinal $|\Gamma|$ of Γ is the least ordinal λ such that $\Gamma^{\lambda+1} = \Gamma^\lambda$. The set defined by Γ is $\Gamma^\infty = \Gamma^{|\Gamma|}$. Γ is monotone if $\Gamma(X) \subseteq \Gamma(Y)$ wherever $X \subseteq Y \subseteq \omega$. For monotone Γ we have

$$\Gamma^\infty = \cap\{X: \Gamma(X) \subseteq X\} .$$

Monotone inductive definitions have long been used in logic and in particular in recursion theory. For example the definitions of the terms, formulas and theorems of predicate logic may be naturally formulated as monotone inductive definitions. More generally Post's production systems give a wide class of monotone inductive definitions, for defining sets of strings of symbols in a finite alphabet. These lead to a natural characterisation of the class of recursively enumerable sets of integers. All these inductive definitions have closure ordinal $\leq \omega$. But inductive definitions with larger closure ordinals may also be considered, and they determine notation systems for ordinals in the following way. For each $x \in \Gamma^\infty$ let $|x|_\Gamma$ be the least ordinal λ such that $x \in \Gamma^{\lambda+1}$. Then $\langle \Gamma^\infty, |\ |_\Gamma \rangle$ is a notation system for the ordinal $|\Gamma|$. Note that because $|\,|_\Gamma$ maps Γ^∞ onto $|\Gamma|$, $|\Gamma|$ must be a countable ordinal. For example let Λ be the i.d.:

$$\Lambda(X) = \{1\} \cup \{2^x: x \in X\} \cup \{3.5^e: \forall n[e](n) \in X\} ,$$

where $[e]$ is the e'th primitive recursive function, in a standard recursive enumeration of them. Then $\langle \Lambda^\infty, |\,|_\Lambda \rangle$ is a slightly modified version of Kleene's system of notations for the recursive ordinals. i.e. Λ^∞ is a complete Π^1_1 set such that $|\Lambda| = \omega_1$, the first non-recursive ordinal. Note that Λ is monotone. Certain monotone i.d.'s are basic to Kleene's definition of recursion in higher type objects, [9] . Also monotone i.d.'s are extensively investigated in [13].

As ω_1 is a constructive analogue of the first uncoutable ordinal, it was natural to try to formulate constructive analogues for larger initial ordinals by constructing systems of notations for them. This led to the use of non-monotone i.d.'s. (See [14] and [15].) An independent development led to the Kripke-Platek theory of recursion on admissible ordinals. These ordinals are a constructive analogue of the regular ordinals, the first two admissible ordinals being ω and ω_1 .
The main aim of this paper is to formulate constructive analogues for large regular ordinals, and to obtain notation systems for them using non-monotone inductive definitions.

Our results will be concerned with classes $\mathcal{C}$ of those i.d.'s that are definable in a certain way. Thus we say that the i.d. Γ is Π^n_m if $\{(x, X) \in \omega \times P(\omega) : x \in \Gamma(X)\}$ is definable by a Π^n_m formula in the language of finite types over arithmetic. Similarily we define the classes of Σ^n_m and Δ^n_m i.d.'s. For example the i.d.'s involved in Post's production systems are all Σ^0_1 when coded on ω . The operator Λ , above, is an example of a Π^0_1 monotone i.d. We shall write Π^n_m-mon for the class of monotone Π^n_m i.d.'s. Similarily for Σ^n_m-mon and Δ^n_m-mon. Given a class $\mathcal{C}$ of i.d.'s we will be interested in $|\mathcal{C}| = \mathrm{Sup}\{|\Gamma| : \Gamma \in \mathcal{C}\}$ and Ind $(\mathcal{C}) = \{X \subseteq \omega : X \leq_m \Gamma^\infty$ for some $\Gamma \in \mathcal{C}\}$. Here $X \leq_m Y$ means that X is many-one reducible to Y .

In many cases $|\mathcal{C}|$ can be compared with $\omega(\mathcal{R})$ for a suitably chosen class $\mathcal{R}$ of relations on ω. $\omega(\mathcal{R})$ is defined to be the sup. of the order types of the well-ordering relations in $\mathcal{R}$. Thus it is well-known that $\omega_1 = \omega(\Delta_1^0) = \omega(\Delta_1^1)$.

We next list some of the earlier results on the ordinals of i.d.'s.

Proposition.

(i) $|\Pi_0^0| = |\Sigma_1^0| = \omega$;

(ii) (Spector [20]) $|\Pi_1^0\text{-mon}| = |\Pi_1^1\text{-mon}| = \omega_1$;

(iii) (Gandy, unpublished) $|\Pi_1^0| = \omega_1$;

(iv) (Richter, [16]) $|\Pi_2^0|$ is a large admissible ordinal; e.g. much larger than the first recursively Mahlo ordinal;

(v) (Putnam, [14]) $|\Delta_2^1| = \omega(\Delta_2^1)$;

(vi) (Gandy, unpublished) $|\Sigma_2^1\text{-mon}| = \omega(\Delta_2^1)$.

We now summarise our results. π_m^n is the least ordinal λ such that $\langle L_\lambda, \in\rangle$ reflects every Π_m^n sentence. σ_m^n is defined using Σ_m^n sentences. For a precise definition see § 1.

Theorem A.

$$|\Pi_m^0| = |\Sigma_{m+1}^0| = \pi_{m+1}^0 = \sigma_{m+2}^0 .$$

Theorem B.

$$|\Pi_1^1| = \pi_1^1 \quad \text{and} \quad |\Sigma_1^1| = \sigma_1^1 .$$

In general the characterisations of the closure ordinals of i.d.'s must be more complicated. If A is is a relation on ordinals let $\pi_m^n(A)$ be the least ordinal λ such that

$\langle L_\lambda[A], \in, A\rangle$ reflects every Π^n_m sentence. Similarily for $\sigma^n_m(A)$. Let $\pi^n_m(\Gamma) = \pi^n_m(A_\Gamma)$ where $A_\Gamma = \{(n, \alpha): \alpha \in ON \;\&\; n \in \Gamma^\alpha\}$. Also let $\sigma^n_m(\Gamma) = \sigma^n_m(A_\Gamma)$.

Theorem C. For $m, n > o$

(i) $|\Pi^n_m| = \pi^n_m(\Gamma)$ where Γ is complete Π^n_m.

(ii) $|\Sigma^n_m| = \sigma^n_m(\Gamma)$ where Γ is complete Σ^n_m.

The proofs of the above characterisations actually give much more information. In each case, as well as characterising $|\mathcal{C}|$ we may also characterise Ind $(\mathcal{C})$. In the next result we use the notion of a "closed" class $\mathcal{C}$. Each Π^n_m and Σ^n_m is a closed class for $m > o$ and every closed class $\mathcal{C}$ has a " -complete" element. See § 8 for a definition of this notion.

Theorem D.

If $\mathcal{C}$ is a closed class $\supseteq \Pi^o_1$, Γ is $\mathcal{C}$-complete and $\lambda = |\mathcal{C}|$ then

(i) λ is admissible relative to A_Γ;

(ii) λ is projectible to ω relative to A_Γ;

i.e. there is a λ-recursive in A_Γ injection $f:\lambda \to \omega$;

(iii) Ind $(\mathcal{C}) = \{X \subseteq \omega : X$ is λ - r.e. relative to $A_\Gamma\}$.

When $\mathcal{C}$ is "sufficiently absolute" then $A_\Gamma \restriction \lambda$ is λ-recursive, so that the relativisation to A_Γ may be omitted in the statement of Theorem D. This is the case when $\mathcal{C}$ is Π^o_{m+1}, Σ^o_{m+2}, Π^1_1 or Σ^1_1.

Results along the lines of theorems C and D have been recently obtained, independently, by Moschovakis. Moreover he has generalised them to classes of inductive definitions on arbitrary abstract structures.

In the next result we locate the ordinals of inductive definitions in relation to the ordinals of certain wellorderings.

Part (i) has been independently obtained by Cenzer (see [5]).

Theorem E. Let $m, n > o$.

(i) If $m+n > 2$ then $|\Delta^n_m| = \omega(\Delta^n_m)$.

(ii)
$$|\Delta^n_{m+1}| \begin{array}{l} > |\Pi^n_m| \geq \pi^n_m \geq \omega(\Sigma^n_m) \geq \\ > |\Sigma^n_m| \geq \sigma^n_m \geq \omega(\Pi^n_m) \geq \end{array} \omega(\Delta^n_m) .$$

Note that $m+n > 2$ is essential in (i) as $|\Delta^1_1| \geq |\Pi^o_2| > \omega_1 = \omega(\Delta^1_1)$. When $m+n > 2$ Sacks has shown in [18] that $\omega(\Delta^n_m)$ is a stable ordinal, so that $|\Delta^n_m|$ is stable. It might be conjectured from this that $|\Delta^1_1|$ is at least admissible. But we have:

Theorem F. $|\Delta^1_1|$ is not admissible.

The diagram in (ii) of theorem E leaves open the order relationship between several pairs of ordinals of the diagram.

The next result, obtained independently by Aanderaa in [1] , gives us some more information.

Theorem G. If $m, n > o$ then

$|\Pi^n_m| \neq |\Sigma^n_m|$.

When $m = n = 1$ this result was first proved by showing directly that $\pi^1_1 \neq \sigma^1_1$. But we do not know if $\pi^n_m \neq \sigma^n_m$ for $n + m > 2$.

The proof of theorem G is symmetric between Π^n_m and Σ^n_m and hence gives no information on the relative magnitudes of the two ordinals.

This is explained by the following result of Aanderaa, which we state here for completeness. (See [1]).
PW($\mathcal{C}$) denotes that $\mathcal{C}$ has the pre-wellordering property. See [1] for a precise definition.

Theorem (Aanderaa). If $m, n > o$ then

(i) $PW(\Pi^n_m) \Rightarrow |\Pi^n_m| < |\Sigma^n_m|$,

(ii) $PW(\Sigma^n_m) \Rightarrow |\Sigma^n_m| < |\Pi^n_m|$.

The following summarises what is known about when the pre-well-ordering property holds.

<u>Proposition</u>.

(i) $PW(\Pi^1_1)$ and $PW(\Sigma^1_2)$,

(ii) $V = L$ implies $PW(\Sigma^1_m)$ for $m > 2$,

(iii) PD implies $PW(\Pi^1_{2m+1})$ and $PW(\Sigma^1_{2m+2})$ for $m > o$.

Here PD denotes the axiom of projective determinacy.
It follows that $|\Pi^1_1| < |\Sigma^1_1|$ and $|\Sigma^1_2| < |\Pi^1_2|$.

This should be compared with the following. (See [21] for more details.)

<u>Proposition</u>.

(i) $\omega(\Sigma^1_1) = \omega(\Delta^1_1) = \omega_1$ and

$\omega(\Pi^1_1) = \omega_1^+$, where α^+ is the first admissible ordinal $> \alpha$;

(ii) $\omega(\Pi^1_2) = \omega(\Delta^1_2)$ and

$\omega(\Delta^1_2) < \omega(\Sigma^1_2) < \omega(\Delta^1_2)^+$;

(iii) $V = L$ implies $\omega(\Delta^1_m) = \omega(\Pi^1_m) < \omega(\Sigma^1_m)$ for $m > 2$.

These results lead to an improvment of Theorem E(ii) in certain cases. For example we have

$\sigma^1_1 = |\Sigma^1_1| > |\Pi^1_1| = \pi^1_1 > |\Delta^1_1| > \omega(\Pi^1_1) > \omega(\Sigma^1_1) = \omega(\Delta^1_1)$,

$|\Pi^1_2| > |\Sigma^1_2| \geq \sigma^1_2 > \omega(\Sigma^1_2) > \omega(\Pi^1_2) = \omega(\Delta^1_2) = |\Delta^1_2|$,

and $|\Pi^1_2| \geq \pi^1_2 > \omega(\Sigma^1_2)$.

Whether $|\Sigma^1_2| = \sigma^1_2$ or $|\Pi^1_2| = \pi^1_2$ remains open. Also, the relationship between π^1_2 and $|\Sigma^1_2|$ or σ^1_2 is not known.

The paper is divided into two parts. In part I we give alternative characterisations of some of the reflecting properties, compare them with the reflecting properties for the indescribable cardinals, and investigate their relative magnitudes. See § 1 for a survey of the definitions and results of Part I. This part makes no use of inductive definitions and may be read without reference to part II.

In part II we prove the results stated in this introduction. Most of this part depends only on § 1 of Part I, so that the reader mainly interested in inductive definitions can probably omit the other sections of part I on a first reading.

In § 7 we examine first order inductive definitions. In particular we give upper bounds to their ordinals, proving half of theorem A. For the other half we need the construction introduced in § 8. In this section we formulate the notion of a closed class of operators. The construction of the notation systems $\mathfrak{M}^{Ⓗ}$ and the associated coding lemma are the key to getting lower bounds for the ordinals of inductive definitions, and to proving theorem D. The coding lemma is proved in the appendix. The proof of theorem A is completed in § 9. Theorems B and C are proved in § 10, while § 11 has proofs of theorems E, F and G. Many of the results in this paper were first announced in [2] .

§ 1. Summary of definitions and results.

In this part we shall study some of the classes of ordinals that will be used to characterise the ordinals associated with inductive definitions. These classes of ordinals will be defined in terms of certain "reflecting properties" closely analogous to those used in defining the indescribable cardinals of Hanf and Scott. (see [7] and also Lévy's [11] for a detailed discussion). In order to bring out this analogy we shall start by considering the indescribable cardinals.

We shall use a perhaps excessively large language $\mathcal{L}$ within which we can conveniently formulate all our reflection properties. $\mathcal{L}$ has the usual propositional connectives, and has variables and quantifiers for all finite types (variables of type 0 range over individuals, those of type 1 range over sets of individuals, etc...). $\mathcal{L}$ also has a name (individual constant) for each set and a name (relation symbol) for each relation on sets. (We will use the same symbol for the object and its name). In particular $\in$ will denote the membership relation between sets. The restricted quantifiers $(\forall x \in y)$, $(\exists x \in y)$ are defined in the usual way. Formulae of $\mathcal{L}$ may be classified according to their prenex form. When doing this we shall follow Lévy in ignoring restricted quantifiers that do not bound unrestricted quantifiers. A formula is Π^n_m (Σ^n_m) if it is logically equivalent to a formula in prenex form which first has m alternating blocks of type n universal and existential quantifiers starting with a block of universal (existential) quantifiers and then has quantifiers of types $< n$ and restricted quantifiers. The allowance for restricted quantifiers is of course only significant when $n = 0$, If a Π^n_m formula φ contains no constants then we call it a Π^{n-}_m formula. Similarily for Σ^{n-}_m .

If $R_1,\dots,R_n,a_1,\dots,a_m$, are the relation symbols and individual constants occurring in a sentence φ of $\mathcal{L}$ let $A \models \varphi$ denote that A is a non-empty set such that $a_1,\dots,a_m \in A$ and φ is true in the structure $\langle A, R_1 \upharpoonright A,\dots,R_n\ A, a_1,\dots,a_m\rangle$.

We can now define the (weak) indescribables.

1.1. Definition. Let $X \subseteq On$ and $\alpha \in On$. If φ is a sentence of $\mathcal{L}$ then <u>α reflects φ on X</u> if

$$\alpha \models \varphi \Longrightarrow (\exists \beta \in X \cap \alpha)\ \beta \models \varphi$$

(Note that On is the class of all ordinal and that we identify an ordinal with its set of predecessors). <u>α reflects φ</u> if α reflects φ on On .

<u>α is Π^n_m (Σ^n_m) indescribable [on X]</u> if α reflects [on X] every Π^n_m (Σ^n_m) sentence of $\mathcal{L}$.

Some properties of the indescribable cardinals are summarised in the following theorems. Proofs of most of these may be found in [11] .

1.2. Theorem. α is Π^0_2-indescribable $\Longleftrightarrow$ $\alpha > \omega$ is regular.

Let $Rg = \{\alpha > \omega \mid \alpha \text{ is regular}\}$. The ordinal α is <u>Mahlo on X</u> if for every $f:\alpha \to \alpha$ there is a $\beta > 0$ closed under f such that $\beta \in X \cap \alpha$.

1.3. Theorem.

(i) the following are equivalent

a) α is Π^0_0-indescribable on X ,

b) α is Σ^0_2-indescribable on X ,

c) $\alpha = \sup(X \cap \alpha)$.

(ii) the following are equivalent

a) α is Π^0_2 indescribable on X

b) α is Π^1_0 indescribable on X

c) α is Mahlo on X

(iii) α is Π^1_n indescribable on $X \Longleftrightarrow \alpha$ is Σ^1_{n+1} indescribable on X.

(iv) If $n>0$ or $m>2$ ($n>0$ or $m>3$) then α is Π^n_m (Σ^n_m) indescribable on $X \Longleftrightarrow \alpha$ is $\Pi^n_m(\Sigma^n_m)$ indescribable on $X \cap Rg$.

Hierarchies of classes of large cardinals have been obtained by iterating such operators as L and M where for $X \subseteq On$:

$$L(X) = \{\alpha \in X \mid \alpha = \mathrm{Sup}\,(X \cap \alpha)\}$$

$$M(X) = \{\alpha \in X \mid \alpha \text{ is Mahlo on } X\} .$$

Iterations of an operator F are defined by transfinite induction on λ:

$$F^\lambda(X) = X \cap \bigcap_{\mu<\lambda} F(F^\mu(X))$$

The elements of $L^\lambda(Rg)$ are the (weak) <u>λ-hyperinaccessibles</u>, while the elements of $M^\lambda(Rg)$ are the (weak) <u>λ-hyperMahlo ordinals</u>. Let $H_1(X) = \{\alpha \in X \mid \alpha$ is Π^0_1-indescribable on $X\}$ and let $H_{n+2}(X) = \{\alpha \in X \mid \alpha$ is Π^1_n-indescribable on $X\}$. Then by theorem 1.3 $H_1 = L$ and $H_2 = M$.

The relative magnitudes of the ordinals in $H_n(Rg)$ may be indicated by using the following diagonalisation of iterations:

$$F^\Delta(X) = \{\alpha>0 \mid \alpha \in F^\alpha(X)\} .$$

1.4. <u>Theorem</u>. (Lévy) If $n>0$ then

$$H_{n+1}(Rg) \subseteq H^\Delta_n(Rg) \;,\; (H^\Delta_n)^\Delta\,(Rg) \;, \text{ etc...}$$

Let us now turn to the strongly indescribable cardinals. These are defined using reflecting properties of the cumulative hierarchy of sets. Let $R(\alpha) = \bigcup_{\beta<\alpha} P(R(\beta))$ for all $\alpha \in On$. ($P(x)$ is the power set of x).

1.5. <u>Definition</u>. <u>$R(\alpha)$ reflects φ on X</u> $\subseteq On$ if

$$R(\alpha) \models \varphi \Longrightarrow (\exists \beta \in X \cap \alpha)\; R(\beta) \models \varphi .$$

<u>$R(\alpha)$ reflects φ</u> if $R(\alpha)$ reflects φ on On.

<u>α is strongly Π^n_m (Σ^n_m) indescribable [on X]</u> if $R(\alpha)$ reflects [on X] every Π^n_m (Σ^n_m) sentence of $\mathcal{L}$.

The properties of the notions of definition 1.5 closely resemble those of definition 1.1. The strong Π^0_2-indescribables coincide with the strongly inaccessible ordinals. For $n>0$ an ordinal is strongly Π^n_m (Σ^n_m) indescribable if and only if it is strongly inaccessible and is Π^n_m (Σ^n_m) indescribable. So, assuming the GCH, the two notions coincide when $n>0$.

Let L_α be the set of constructible sets of order $<\alpha$. (i.e. $L_\alpha = \bigcup_{\beta<\alpha} \mathrm{Def}(L_\beta)$ where $\mathrm{Def}(x)$ is the set of subsets of x definable in $\langle x, \in\restriction x, a\rangle_{a\in x}$).

1.6. <u>Definition</u>. <u>L_α reflects φ on X</u> if

$$L_\alpha \models \varphi \implies (\exists \beta \in X \cap \alpha)\ L_\beta \models \varphi .$$

<u>L_α reflects</u> φ if L_α reflects φ on On.

If this definition is used as in definition 1.5 the resulting indescribability notions may easily be seen to coincide with those of definition 1.1.

In order to obtain the classes of ordinals that we are interested in we restrict the language $\mathcal{L}$. Let $\mathcal{L}_\in$ be the sub-language of $\mathcal{L}$ obtained by only allowing $\in$ as a relation symbol.

1.7. <u>Definition</u>.

<u>α is Π^n_m (Σ^n_m) reflecting [on X]</u> if L_α reflects [on X] every Π^n_m (Σ^n_m) sentence of $\mathcal{L}_\in$.

Some properties of this definition are summarised in the following theorems, which should be compared with theorems 1.1 and 1.2.

1.8. Theorem. α is Π^0_2 - reflecting iff α is an admissible ordinal $> \omega$. This result and theorem 1.9 below will be proved in § 2.

Let $Ad = \{\alpha > \omega \mid \alpha \text{ is admissible}\}$. $\alpha \in Ad$ is recursively Mahlo if for every α - recursive function $f : \alpha \to \alpha$ there is an ordinal $\beta > 0$ closed under f such that $\beta \in X \cap \alpha$.

1.9. Theorem.

(i) The following are equivalent

a) α is Π^0_0-reflecting on X

b) α is Σ^0_2-reflecting on X

c) $\alpha = \mathrm{Sup}(X \cap \alpha)$

(ii) α is Π^0_2-reflecting on $X \Longleftrightarrow \alpha$ is recursively Mahlo on X.

(iii) α is Π^0_n-reflecting on $X \Longleftrightarrow \alpha$ is Σ^0_{n+1}-reflecting on X.

(iv) If $n > 0$ or $m > 2$ ($n > 0$ or $m > 3$) then α is $\Pi^n_m(\Sigma^n_m)$ reflecting on $X \Longleftrightarrow \alpha$ is $\Pi^n_m(\Sigma^n_m)$ reflecting on $X \cap Ad$.

As it is often easier to work with ordinals rather than the constructible hierarchy the following characterisations will be useful. Let $\mathcal{L}_p$ be the sublanguage of $\mathcal{L}$ that has relation symbols only for the primiteve recursive relations on sets (see [8] for the properties of this notion).

1.10. Theorem. α is $\Pi^n_m(\Sigma^n_m)$ reflecting [on X] if and only if α reflects [on X] every $\Pi^n_m(\Sigma^n_m)$ sentence of $\mathcal{L}_p$.

The primitive recursive relations in the language $\mathcal{L}_p$ are needed for reflecting properties on ordinals in order to compen-

sate for the richness of the $\in$ relation for reflecting properties on the constructible hierarchy.
Theorem 1.10. will be proved in § 3 .

$L^{\lambda}(Ad)$ is the class of <u>λ-recursively inaccessible ordinals</u>, while if $RM(X) = \{\alpha \in X \mid \alpha$ is recursively Mahlo on $X\}$ then $RM^{\lambda}(Ad)$ is the class of <u>λ-recursively Mahlo</u> ordinals. Let $M_n(X) = \{\alpha \in X \mid \alpha$ is Π^0_n reflecting on $X\}$. Then $M_0 = M_1 = L$ and $M_2 = RM$. The next result indicates the relative magnitudes of the ordinals in $M_n(Ad)$ and should be compared with theorem 1.4.

1.11. <u>Theorem</u>. If $n > 0$ then

$$M_{n+1}(Ad) \subseteq M_n^{\Delta}(Ad),\ (M_n^{\Delta})^{\Delta}(Ad),\ \text{etc.} \ .....$$

This will be proved in § 4 .

1.12. <u>Definition</u>. Let $\pi^n_m(\sigma^n_m)$ be the least $\Pi^n_m(\Sigma^n_m)$ reflecting ordinal.

By 1.9. $\pi^0_0 = \pi^0_1 = \omega$ and $\pi^0_2 = \omega_1$ are the recursive analognes of the first two regular cardinals. What can we say about π^0_3? By 1.9. and 1.11. π^0_3 is greater than the least recursively Mahlo ordinal, the least recursively hyper-Mahlo ordinal etc. In fact π^0_3 appears to be greater than any "reasonable" iteration into the transfinite of this diagonalisation process. When one thinks of a corresponding cardinal in set theory (with "recursively Mahlo" now replaced by "Mahlo") the cardinal which comes to mind is the least Π^1_1-indescribable cardinal. We shall now try and justify the view that Π^0_3-reflection is the recursive analogue of Π^1_1-indescribability. The same ideas with some additional notational complexity provide an analogy between Π^0_{n+2}-reflection and Π^1_n-indescribability for all $n > 0$, but we shall concentrate on the case $n = 1$.

The analogy is obtained as follows. A class of cardinals, called the 2-regular cardinals, is defined, as well as a recursive analogue of this class whose members are called 2-admissible. We then show that a cardinal is 2-regular if and only if it is strongly Π^1_1-indescribable, and an ordinal is 2-admissible if and only if it is Π^0_3-reflecting.

Certain properties of infinity can be stated in terms of fixed points of operations. For example $\kappa > \omega$ and κ is regular if and only if:

(1) for every $f : \kappa \to \kappa$ there is some $0 < \alpha < \kappa$ such that $f''\alpha \subseteq \alpha$. (We say α is a <u>witness</u> for f .)

If we modify (1) by requiring that the witness be regular, we obtain the hyper-Mahlo cardinals, etc.

An alternative way of modifying (1) is by using higher type operations on κ .

Let $F : {}^{\kappa}\kappa \to {}^{\kappa}\kappa$. F is κ-<u>bounded</u> if for every $f : \kappa \to \kappa$ and $\xi < \kappa$, the value $F(f)(\xi)$ is determined by less than κ values of f . More precisely, F is κ-bounded if

$$\forall f \exists \gamma < \kappa \, \forall g [g \restriction \gamma = f \restriction \gamma \implies F(f) = F(g)] .$$

$0 < \alpha < \kappa$ is a <u>witness for</u> F if for every $f : \kappa \to \kappa$,

$$f''\alpha \subseteq \alpha \implies F(f)''\alpha \subseteq \alpha .$$

1.3 <u>Definition</u>. $\kappa > 0$ is 2-<u>regular</u> if every κ-bounded $F : {}^{\kappa}\kappa \to {}^{\kappa}\kappa$ has a witness.

1.14. <u>Theorem</u>. $\varkappa$ is 2-regular iff $\varkappa$ is strongly Π_1^1-indescribable.

We now look at a recursive analogue of 2-regularity. Roughly speaking the following definition of 2-admissible is obtained by replacing in the definition of 2-regular, "bounded" by "recursive" and the functions by their Gödel numbers. In the following definition we write $\{\xi\}_\varkappa : \varkappa \to \varkappa$ to mean that $\{\xi\}_\varkappa$ is total on $\varkappa$.

1.15. <u>Definition</u>. (i) Let $\varkappa \in \mathrm{Ad}$ and $\xi < \varkappa$. $\{\xi\}_\varkappa$ <u>maps $\varkappa$-recursive functions to $\varkappa$-recursive functions if</u>

$$\forall \beta < \varkappa[\{\beta\}_\varkappa : \varkappa \to \varkappa \implies \{\{\xi\}_\varkappa(\beta)\}_\varkappa : \varkappa \to \varkappa] ;$$

(ii) Suppose $\{\xi\}_\varkappa$ maps $\varkappa$-recursive functions to $\varkappa$-recursive functions. $\alpha \in \varkappa \cap \mathrm{Ad}$ is a <u>witness for</u> ξ if $\xi < \alpha$ and $\{\xi\}_\alpha$ maps α-recursive functions to α-recursive functions.

(iii) $\varkappa \in \mathrm{Ad}$ is 2-admissible if every $\xi < \varkappa$ such that $\{\xi\}_\varkappa$ maps $\varkappa$-recursive functions to $\varkappa$-recursive functions has a witness.

1.16. <u>Theorem</u>. $\varkappa$ is 2-admissible iff $\varkappa$ is Π_3^0-reflecting.

Theorems 1.14 and 1.16 will be proved in § 5.

Certain classes of ordinals, defined in terms of reflecting properties, also have characterisations in terms of stability properties. Let $A \prec_{\Sigma_1^0} B$ if A and B are transitive sets such that $A \subseteq B$ and $B \models \varphi \implies A \models \varphi$ for every Σ_1^0 sentence φ of $\mathcal{L}_\in$ that only has constants for elements of A. Kripke has defined the notion of an ordinal α being β-stable (see [10]).

His definition used his systems of equations for defining recursion on ordinals. For admissible β he gave the following characterisation, which we shall take as a definition:

1.17. Definition. α is β-stable if $\alpha < \beta$ and $L_\alpha \prec_{\Sigma_1^0} L_\beta$.

When β is not admissible, this notion may well diverge from Kripke's original one.

1.18. Theorem.

α is Π_0^1-reflecting if and only if α is $\alpha+1$-stable.

1.19. Theorem. For countable α

α is Π_1^1-reflecting if and only if α is α^+-stable where α^+ is the first admissible ordinal $> \alpha$.

These results will be proved in § 6 .

Given $A \subseteq {}^n\mathrm{ON}$ all of our definitions and results will relativise to A . As we shall need the relativisations in Part II we spell out exactly what this means.

Definition 1.6. is relativised by using $\langle L_\alpha[A] : \alpha \in \mathrm{ON}\rangle$ instead of $\langle L_\alpha : \alpha \in \mathrm{ON}\rangle$. Here $L_\alpha[A] = \bigcup_{\beta<\alpha} \mathrm{Def}_A(L_\beta[A])$ where $\mathrm{Def}_A(x)$ is the set of subsets of x definable in $\langle x, \in\restriction x, A\restriction x, a\rangle_{a\in x}$. The language $\mathcal{L}_\in$ must be replaced by the language $\mathcal{L}_\in(A)$ which is $\mathcal{L}_\in$ with an added n-ary relation symbol to denote A . Definition 1.7. becomes α is <u>$\Pi_m^n(A)$-reflecting [on X]</u> if $L_\alpha[A]$ reflects [on X] every Π_m^n sentence of $\mathcal{L}_\in(A)$. Similarily for $\Sigma_m^n(A)$-reflecting.

Theorems 1.8. and 1.9. relativise in the obvious way. Ad must be replaced by $\mathrm{Ad}(A) = \{\alpha > \omega \mid \alpha$ is admissible relative to $A\restriction\alpha\}$.

The language $\mathcal{L}_p(A)$ is defined by allowing relation symbols for all relations primitive recursive in A . Most of the proofs relativise in a routine way.

§ 2. Elementary facts.

In order to prove our theorems we shall need to assume some familiarity with the notions of primitive recursive set function, admissible class, admissible ordinal and ordinal recursion on an admissible ordinal. We shall use [8] as our basic reference and will usually follow the terminology they use. We shall also need to refer to [6] when we use Jensen's notion of a rudimentary set function.

The notion of a primitive recursive function with domain M has been formulated for various classes M e.g. ${}^{m}\omega$, ${}^{m}ON$ and ${}^{m}V$. As shown in [8] all these notions turn out to be special cases of the following: $F : M \to V$ is primitive recursive if M is a primitive recursive class and F is the restriction to M of a primitive recursive set function. In [8] a transitive prim closed class M is defined to be admissible if M satisfies the Σ_1^0-collection principle (there called Σ_1^0-reflection principle) which we shall formulate as follows:

For every prenex Σ_1^0 formula θ of $\mathcal{L}_\in$ if $M \models \forall x \in a\theta$ then $M \models \forall x \in a\theta^b$ for some $b \in M$, where if θ is $\exists y_1 \ldots \exists y_k \Psi$, with $\Psi \Sigma_0^0$, then θ^b is $\exists y_1 \in b \ldots \exists y_k \in b\Psi$.

We shall find it more useful to use the characterization in [6].

2.1. Definition. The transitive class M is admissible if M is rud closed and satisfies Σ_1^0-collection.

This definition is relativized by replacing Σ_1^0-collection by $\Sigma_1^0(A)$-collection, obtained by using $\mathcal{L}_\in(A)$ instead of $\mathcal{L}_\in$, and adding the condition that $a \in M \Rightarrow A \cap a \in M$.

A relation R on a transitive set M is Σ_1^0 on M if

R is defined on M by a Σ_1^0 formula of $\mathcal{L}_\in$. A partial function with arguments and values in M is Σ_1^0 on M if its graph is. We shall assume some familiarity with the closure properties of these relations and functions on an admissible M, as presented for example in [8]. In particular we shall need the following:

2.2. Proposition. (Definition by Σ_1^0-recursion). Let M be an admissible set. Let G be a function such that $G \upharpoonright M : M \times M \to M$ and $G \upharpoonright M$ is Σ_1^0 on M. Let

$$F(x) = G(x, F \upharpoonright x).$$

Then $F \upharpoonright M : M \to M$ and $F \upharpoonright M$ is Σ_1^0 on M. Moreover the Σ_1^0 definition of $F \upharpoonright M$ depends only on the Σ_1^0 definition of $G \upharpoonright M$ (and not on M).

Usually we will only be interested in $F \upharpoonright M \cap ON$.

For the notion of an admissible ordinal α and α-recursion we shall follow [8]. An ordinal α is admissible if L_α is admissible. $f : {}^n\alpha \to \alpha$ is α-recursive if it is Σ_1^0 on L_α.

The following lemma will be useful and the proof will illustrate some of the techniques of α-recursion.

2.3. Lemma. If $\alpha > \omega$ is an admissible ordinal and $f : \alpha \to \alpha$ is α-recursive then there are arbitrarily large limit ordinals $< \alpha$ that are closed under f.

Proof. Let $\alpha > \omega$ be admissible and let $f : \alpha \to \alpha$ be α recursive. Define $g : \alpha \to \alpha$ by $g(x) = \mathrm{Max}(x+1, \sup_{y \leq x} f(y))$. Then g is α-recursive, $x < g(x)$ and $f(x) \leq g(y)$ for $x \leq y < \alpha$. Given $\gamma_0 < \alpha$ let $\gamma_n = g^n(\gamma_0)$. Then $\gamma_0 < \gamma_1 < \ldots < \alpha$ and $x \leq \gamma_n \Longrightarrow f(x) \leq \gamma_{n+1}$. Let $\gamma = \sup_{n<\omega} \gamma_n$. Then $\gamma \leq \alpha$ is a limit ordinal such that $\gamma_0 < \gamma$ and γ is closed under f as

$$x < \gamma \Longrightarrow x \leq \gamma_n \text{ for some } n$$
$$\Longrightarrow f(x) \leq \gamma_{n+1} < \gamma .$$

So it only remains to show that $\gamma < \alpha$. For this we need 2.2. Let $F(x) = G(x, F \restriction x)$ where $G(x,y) = g(z)$ if x is a successor ordinal, y is a function such that $y(x-1)$ is defined with value $z < \alpha$, and $G(x,y) = \gamma_0$ otherwise.

Then it is not hard to see that $\gamma_n = F(n)$ for each $n \in \omega$, and that as $G \restriction L_\alpha : L_\alpha \times L_\alpha \to L_\alpha$ and is Σ_1^0 on L_α it follows that $F \restriction \alpha$ is α-recursive and hence $\gamma = \operatorname{Sup}_{n<\omega} \gamma_n = \operatorname{Sup}_{n<\omega} F(n) < \alpha$.

<u>Proof of theorem 1.8.</u>

Let α be Π_2^0-reflecting. If $a < \alpha$ then $L_\alpha \models \neg(a \in a)$. Hence there is a $\beta < \alpha$ such that $L_\beta \models \neg(a \in a)$; i.e. $a < \beta < \alpha$. Hence α is a limit number. So $L_\alpha \models \forall x \exists y(x \in y)$, which implies that there is a $\beta < \alpha$ such that $L_\beta = \forall x \exists y(x \in y)$. Hence α is a limit number $> \omega$. Using lemma 6 of [6] it is not hard to show that L_α is rud closed for any limit ordinal α. Hence it remains only to show that L_α satisfies Σ_1^0-collection. So let $L_\alpha \models \forall x \in a\theta$ where θ is a Σ_1^0 formula of $\mathcal{L}_\in$. Then by Π_2^0-reflection there is a $\beta < \alpha$ such that $L_\beta \models \forall x \in a\theta$. Now if $b = L_\beta \in L_\alpha$ then $L_\alpha \not\models \forall x \in a\theta^b$ as required.

Conversely, let $\alpha > \omega$ be admissible, and let φ be a Π_2^0 sentence of $\mathcal{L}_\in$ such that $L_\alpha \models \varphi$. We may assume that φ has the form $\forall x_1 \ldots x_n \exists y_1 \ldots y_m \Psi$ where Ψ is Σ_0^0. Hence $L_\alpha \models \forall x_1 \ldots x_n \exists y \theta$ where θ is the Σ_0^0 formula $\exists y_1 \in y \ldots \exists y_m \in y \Psi$. For simplicity we shall just consider the case when $n = 1$. If $\beta < \alpha$ and $a = L_\beta$ then $L_\alpha \models \forall x_1 \in a \exists y \theta$. Hence by Σ_1^0-collection there is a $b \in L_\alpha$ such that $L_\alpha \models \forall x_1 \in a\, \exists y \in b\theta$. But $b \subseteq L_\gamma$ for some $\gamma < \alpha$ so that $\models \forall x_1 \in a\, \exists y \in L_\gamma \theta$. Let $f(\beta)$ be the least such $\gamma < \alpha$. Then $f: \alpha \to \alpha$ is α-recursive. Let $\beta_0 < \alpha$ such that every constant of θ occurs in L_{β_0}. Then by the lemma 2.3 choose a limit ordinal β such that $\beta_0 < \beta < \alpha$ and β is closed under f. Then we must have $L_\beta \models \forall x_1\, \exists y \theta$ so that α reflects the Π_2^0 sentence φ.

In order to prove (iv) of theorem 1.9 we shall need

2.4. <u>Theorem</u>. There is a Π_3^{0-} sentence σ_0 of $\mathcal{L}_\in$ such that the transitive class M is admissible if and only if $M \models \sigma_0$.

<u>Proof</u>. By lemma 6 of [6] there are binary rud functions $F_0, \ldots, F_8$ such that the class M is rud closed if and only if it is closed under $F_0, \ldots, F_8$.

By lemma 2 of [6] there are Σ_0^{0-} formulae $\varphi_i(x,y,z)$ of $\mathcal{L}_\in$ that define the graphs of F_i for $i \leq 8$. So M is rud closed if and only if $M \models \theta_0$ where θ_0 is the Π_2^{0-} sentence $\bigwedge_{i \leq 8} \forall x\, \forall y\, \exists z \varphi_i(x,y,z)$.

By lemma 9 of [6] we may prove:

2.5. <u>Lemma</u>. There is a Σ_1^{0-} formula $Sat(x,y)$ of $\mathcal{L}_\in$ such that if $\theta(x)$ is a Σ_1^0 formula of $\mathcal{L}_\in$ with x as only free variable and $a = \ulcorner\theta(x)\urcorner$ then for all rud closed M, if the constants of $\theta(x)$ are in M then $a \in M$ and

$$M \models \forall x(\theta(x) \leftrightarrow Sat(a,x)).$$

Using this lemma we see that the transitive rud closed class M is admissible if and only if $M \models \theta_1$ where θ_1 is the Π_3^{0-} sentence $\forall u\, \forall v[\forall x \in u\, Sat(v,x) \to \exists z\, \forall x \in u\, Sat(v,x)^z]$. The theorem now follows if we let σ_0 be $\theta_0 \wedge \theta_1$.

Proof of theorem 1.9.

(i) b) $\Longrightarrow$ a) is trivial.

c) $\Longrightarrow$ b). Let $\alpha = \mathrm{Sup}(X \cap \alpha)$ and let φ be a Π_1^0 sentence such that $L_\alpha \models \varphi$. If $a_1, \ldots, a_n$ are the individual constants occurring in φ then $a_1, \ldots, a_n \in L_\alpha$ so that there is a $\beta \in X \cap \alpha$ such that $a_1, \ldots, a_n \in L_\beta$ and hence $L_\beta \models \varphi$, as φ is Π_1^0. So α is Π_1^0 reflecting on X and hence Σ_2^0 reflecting on X, by (iv)

a) $\Longrightarrow$ c). Let α be Π_0^0 reflecting on X and let $\beta < \alpha$. Let φ be the sentence $\neg(\beta \in \beta)$. Then $L_\alpha \models \varphi$, so that as φ is Π_0^0 there is a $\gamma \in X \cap \alpha$ such that $L_\gamma \models \varphi$ and hence $\beta < \gamma$. Hence $\alpha = \mathrm{Sup}(X \cap \alpha)$.

(ii) $\Longrightarrow$. Let α be Π_2^0-reflecting on X. Then α is Π_2^0-reflecting and hence by theorem 1.8 $\alpha \in \mathrm{Ad}$. Now let $f : \alpha \to \alpha$ be α-recursive. Let $\theta(x,y)$ be a Σ_1^0 formula of $\mathscr{L}_\epsilon$ that defines the graph of f on L_α. Then $L_\alpha \models \forall x \exists y(\theta(x,y) \vee (\neg \mathrm{On}(x) \wedge y = 0))$. Hence there is a $\beta \in X \cap \alpha$ such that $L_\beta \models \forall x \exists y(\theta(x,y) \vee (\neg \mathrm{On}(x) \wedge y = 0))$. Hence $\beta > 0$ is closed under f. So α is recursively Mahlo on X.

(ii) $\Longleftarrow$ Let α be recursively Mahlo on X and let φ be a Π_2^0 sentence of $\mathscr{L}_\epsilon$ such that $L_\alpha \models \varphi$. As in the proof of theorem 1.8 we will suppose that φ has the form $\forall x_1 \exists y \theta$ where θ is Σ_0^0 and define the α-recursive function $f ; \alpha \to \alpha$, and the ordinal $\beta_0 < \alpha$. Now let $g(x) = \mathrm{Max}(\beta_0, x+1, f(x))$. Then $g : \alpha \to \alpha$ is α-recursive so that there is a $\beta \in X \cap \alpha$ such that $\beta > 0$ is closed under g. From the definition of g it follows that $\beta > \beta_0$ is a limit ordinal and is closed under f so that $L_\beta \models \forall x_1 \exists y \theta$. Thus α reflects φ as required.

(iii) $\Leftarrow$ is trivial. So for the converse let α be Π^0_n reflecting on X and let φ be Σ^0_{n+1} such that $L_\alpha \models \varphi$. φ has the form $\exists x_1 \ldots \exists x_n \theta(x_1,\ldots,x_n)$ where $\theta(x_1,\ldots,x_n)$ is Π^0_n. So there are $a_1,\ldots,a_n \in L_\alpha$ such that $L_\alpha \models \theta(a_1,\ldots,a_n)$. Hence there is a $\beta \in X \cap \alpha$ such that $L_\beta \models \theta(a_1,\ldots,a_n)$. But then $L_\beta \models \exists x_1 \ldots \exists x_n \theta(x_1 \ldots x_n)$. So α is Σ^0_{n+1} reflecting on X.

(iv) $\Leftarrow$ is trivial as $X \cap Ad \subseteq X$. For the converse we use the Π^{0-}_3 sentence σ_0 given by theorem 2.4. Let θ_0 be a Σ^{0-}_1 sentence expressing the existence of an infinite set. Then $\alpha \in Ad \iff L_\alpha \models \sigma_0 \wedge \theta_0$, and $\sigma_0 \wedge \theta_0$ is a Π^{0-}_3 sentence of $\mathcal{L}_\in$. Now let α be $\Pi^n_m(\Sigma^n_m)$ reflecting on X and let φ be a $\Pi^n_m(\Sigma^n_m)$sentence of $\mathcal{L}_\in$ such that $L_\alpha \models \varphi$. Then because of the restrictions on n and m $\varphi \wedge \sigma_0 \wedge \theta_0$ is a $\Pi^n_m(\Sigma^n_m)$ sentence of $\mathcal{L}_\in$ such that $L_\alpha \models \varphi \wedge \sigma_0 \wedge \theta_0$. Hence there is a $\beta \in X \cap \alpha$ such that $L_\beta \models \varphi \wedge \sigma_0 \wedge \theta_0$. Hence $\beta \in Ad$ so that α reflects φ on $X \cap Ad$ showing that α is $\Pi^n_m(\Sigma^n_m)$ reflecting on $X \cap Ad$.

§ 3. Ordinal theoretic characterisations.

Let us call α $\Pi^n_m(\Sigma^n_m)$ *-reflecting [on X] if α reflects [on X] every $\Pi^n_m(\Sigma^n_m)$ sentence of $\mathcal{L}_p$. Our proof of theorem 1.10 will be a little indirect, in that we first prove theorems 1.8* and 1.9*, obtained from theorems 1.8 and 1.9 by replacing 'reflecting' everywhere by '*-reflecting'. But first we need the following lemma.

3.1. Lemma. There is a bijective primitive recursive function $N : ON \to L$ such that if $(\forall x < \alpha) 2^x < \alpha$ then $L_\alpha = N''\alpha$.

Proof. In lemma 3.2. of [8] a primitive recursive bijection $N : ON \to L$ is obtained from Gödel's primitive recursive surjection $F : ON \to L$ by successively removing repetitions in the values of F . Examining their definition of N it is not hard to see that $N''\alpha = F''\alpha$ for all limit ordinals α . If $(\forall x < \alpha) 2^x < \alpha$ then either $\alpha = 0$, $\alpha = \omega$ or α has the form $\alpha = \epsilon_\beta$. Clearly $L_0 = \emptyset = N''0$. If $\alpha = \omega$ or $\alpha = \epsilon_\beta$ then in [12] it is shown that $L_\alpha = F''\alpha$ and hence it follows that $L_\alpha = N''\alpha$ as α is a limit ordinal.

This result relativises to give a bijection $N_A : ON \to L[A]$ which is primitive recursive in A , such that $L_\alpha[A] = N_A''\alpha$ if $(\forall x < \alpha) 2^x < \alpha$.

3.2. Lemma. If α is Π^0_2 *-reflecting then

(i) α is a limit ordinal $> \omega$.

(ii) $a, b < \alpha \implies a+b < \alpha$.

(iii) $b < \alpha \implies 2^b < \alpha$.

Proof. First note that the graphs of primitive recursive functions are primitive recursive relations and hence are allowed in the language $\mathcal{L}_p$.

(i) If $a < \alpha$ then $\alpha \models \exists x(x=a)$ so that $\beta \models \exists x(x=a)$ for some $\beta < \alpha$ and hence $a < \beta < \alpha$. thus α is a limit number. So $\alpha \models \forall x\, \exists y(x<y)$. By reflection there is a limit ordinal $< \alpha$; i.e. α is a limit ordinal $> \omega$.

(ii) Given $a < \alpha$ we prove that $a+b < \alpha$ by induction on $b < \alpha$. If $b = 0$ this is trivial. If $0 < b < \alpha$ and $\forall x < b$ $a+x < \alpha$ then, as α is a limit ordinal, $\alpha \models \forall x\, \exists y\ulcorner x < b \rightarrow a+x+1 = y\urcorner$. So by reflection $\forall x < b \quad a+x+1 < \beta$ for some $\beta < \alpha$. Hence $a+b = \sup\limits_{x<b} a+x+1 \leq \beta < \alpha$.

(iii) We prove that $2^b < \alpha$ by induction on $b < \alpha$. If $b = 0$ then $2^b = 1 < \omega < \alpha$. If $0 < b < \alpha$ and $\forall x < b\, 2^x < \alpha$ then by (ii) $\forall x < b\, 2^{x+1} = 2^x+2^x < \alpha$. Hence $\alpha \models \forall x \exists y[x < b \rightarrow 2^{x+1} = y]$. So by reflection $\forall x < b\ 2^{x+1} < \beta$ for some $\beta < \alpha$. Hence $2^b = \sup\limits_{x<b} 2^{x+1} \leq \beta < \alpha$.

3.3 Theorem 1.8*

α is Π_2^0 *-reflecting $\iff$ $\alpha \in Ad$.

Proof. Let α be Π_2^0 *-reflecting. Then by (i) of lemma 3.2. $\alpha > \omega$ and α is a limit ordinal so that as we have already observed L_α is rud closed. Hence it suffices to show that L_α satisfies Σ_1^0-collection. So let $L_\alpha \models \forall x \in a\, \exists y\ \Psi(x,y,b)$ where $\Psi(x,y,z)$ is Σ_0^{0-}. (We can assume without loss that there is only one existential quantifier $\exists y$ and only one constant b.) We must find $c \in L_\alpha$ such that

(*) $L_\alpha \models \forall x \in a\ \exists y \in c\ \Psi(x,y,b)$.

Let $R \subseteq^3 ON$ be the primitive recursive relation given by

$R(\alpha,\beta,\gamma) \iff \models \Psi(N(\alpha),N(\beta),N(\gamma))$

Let $a = N(\alpha_o)$, $b = N(\beta_o)$. By the previous lemmas $L_\alpha = N''\alpha$ so that

$$\alpha \models \forall x(R_\in(x,\alpha_o) \to \exists y R(x,y,\beta_o)) \wedge \forall x\ \exists y\ (2^x = y)$$

where $R_\in(\alpha,\beta) \iff N(\alpha) \in N(\beta)$.

Hence, as α is Π^o_2 *-reflecting, there is a $\beta < \alpha$ such that $\beta \models \forall x(R_\in(x,\alpha_o) \to \exists y R(x,y,\beta_o)) \wedge \forall x\ \exists y (2^x = y)$. As $\forall x < \beta(2^x < \beta)$ $L_\beta = N''\beta$ so that

$$L_\beta \models \forall x \in a\ \exists y\ \Psi(x,y,b)$$

(*) follows if we let $c = L_\beta \in L_\alpha$.

<u>3.4. Theorem</u> 1.9 *.

<u>Proof</u>. This follows the same pattern as the proof of theorem 1.9 and so will be omitted. In the proof of (iv) we need the next lemma, which replaces theorem 2.4.

3.5 <u>Lemma</u>. There is a Π^{o-}_3 sentence σ_1 of $\mathcal{L}_p$ such that α is admissible if and only if $\alpha \models \sigma_1$.

<u>Proof</u>. Let us assume that the Π^{o-}_3 sentence σ_o of $\mathcal{L}_\in$ given in theorem 2.4. is in Prenex form with Σ^{o-}_o matrix $\Psi(x_1,\dots,x_k)$. Now let $R(\alpha_1,\dots,\alpha_k) \iff \models \Psi(N(\alpha_1),\dots,N(\alpha_k))$ for $\alpha_1,\dots,\alpha_k \in ON$. Then R is a primitive recursive relation. Let θ_o be obtained from σ_o by replacing $\Psi(x_1,\dots,x_k)$ by $R(x_1,\dots,x_k)$. Then if $L_\alpha = N''\alpha$

$$\alpha \models \theta_o \iff L_\alpha \models \sigma_o .$$

Hence by lemma 3.1. we can let σ_1 be $\theta_0 \wedge \forall x\, \exists y(2^x = y)$.

We can now turn to the proof of theorem 1.10. (i) - (iii) of theorems 1.9 and 1.9* yield the theorem in the cases Π^0_m $(m \leq 2)$ and Σ^0_m $(m \leq 3)$. By theorems 1.8 and 1.8* and (iv) of theorems 1.9 and 1.9* the remaining cases need only be proved when $\alpha \in \mathrm{Ad}$ and $X \subseteq \mathrm{Ad}$. With these restrictions the remaining cases will follow from:

3.6. Lemma. Let $n+m > 0$

(i) For each Π^n_m sentence θ of $\mathcal{L}_p$ there is a Π^n_m sentence θ_ϵ of $\mathcal{L}_\epsilon$ such that for admissible α

$$\alpha \models \theta \iff L_\alpha \models \theta_\epsilon .$$

(ii) For each Π^n_m sentence θ of $\mathcal{L}_\epsilon$ there is a Π^n_m sentence θ_p of $\mathcal{L}_p$ such that for admissible α

$$L_\alpha \models \theta \iff \alpha \models \theta_p .$$

Using this lemma let us conclude the proof of theorem 1.10. Let $n > 0$ or $m > 2$ and let α be Π^n_m reflecting on X . Let θ be a Π^n_m sentence of $\mathcal{L}_p$ such that $\alpha \models \theta$. Then θ_ϵ is a Π^n_m sentence of $\mathcal{L}_\epsilon$ such that $L_\alpha \models \theta_\epsilon$ as α is admissible. Hence there is a $\beta \in X \cap \alpha$ such that $L_\beta \models \theta_\epsilon$. As $X \subseteq \mathrm{Ad}$, β is admissible so that $\beta \models \theta$. Hence L_α reflects θ . Similarly if ($n > 0$ or $m > 3$) and α is Σ^n_m reflecting on X and θ is a Σ^n_m sentence of $\mathcal{L}_p$ then $\neg\theta$ is a Π^n_m sentence of $\mathcal{L}_p$ so that $\neg(\neg\theta)_\epsilon$ is a Σ^n_m sentence of $\mathcal{L}_\epsilon$ and the argument is as above. The proof of the converse implications is exactly similar using (ii) of the lemma instead of (i).

Proof of lemma 3.6.

(i) By the stability theorem 2.5 of [8] we may easily associate with each primitive recursive relation R a Σ_1^{0-} formula $\varphi_R(x_1,\ldots,x_n)$ of $\mathcal{L}_\epsilon$ such that for admissible α and $\alpha_1,\ldots,\alpha_n < \alpha$

$$R(\alpha_1,\ldots,\alpha_n) \iff L_\alpha \models \varphi_R(N(\alpha_1),\ldots,N(\alpha_n)) .$$

Now let θ be a sentence of $\mathcal{L}_p$. If θ contains individual constants for sets that are not ordinals, then $\alpha \models \theta$ can never hold, so let θ_ϵ be $(1 \in 0)$. Otherwise define θ_* as follows. First replace each constant for an ordinal β by a constant for $N(\beta)$. Then replace each occurrence of a relation symbol $R(s_1,\ldots,s_n)$ in θ by $\varphi_R(s_1,\ldots,s_n)$. Then for admissible α it is clear that

$$\alpha \models \theta \iff L_\alpha \models \theta_*$$

Now if θ is Π_m^n and $n > 0$ then θ_* is also Π_m^n and so we can let θ_ϵ be θ_* .

If θ is Π_m^0 $(m > 0)$ then we have to be more careful. We may assume that θ is in prenex form. So it has the form of an alternating sequence of m blocks of universal and existential type 0 quantifiersfollowed by a Π_0^0 formula $\Psi(x_1,\ldots,x_k)$. Now $\Psi(x_1,\ldots,x_k)$ is built up from primitive recursive relations and ordinals using the boolean operations and restricted quantifiers. Hence there is a primitive recursive relation R and ordinals $\beta_1,\ldots,\beta_l$ such that for all α

$$\alpha \models \Psi(\alpha_1,\ldots,\alpha_k) \iff \alpha \models R(\beta_1,\ldots,\beta_l,\alpha_1,\ldots,\alpha_k)$$

Now define θ_ϵ as follows: If m is even, replace $\Psi(x_1,\ldots,x_k)$

in θ by $\varphi_R(N(\beta_1),\ldots,N(\beta_l),x_1,\ldots,x_k)$ and if m is odd, replace $\Psi(x_1,\ldots,x_k)$ in θ by $\neg\varphi_{\neg R}(N(\beta_1),\ldots,N(\beta_l),x_1,\ldots,x_k)$. Then $\theta_\in$ is Π^0_m and has the desired properties.

(ii) Let θ be a sentence of $\mathcal{L}_\in$. If θ contains constants for non-constructible sets, then $L_\alpha \models \theta$ never holds so we can let θ_p be $(0 = 1)$. Otherwise define θ_o as follows. First replace each individual constant for the set a by the constant for ordinal α such that $N(\alpha) = a$. Then replace each occurrence of $s \in t$ in θ by $R_\in(s,t)$, where $R_\in(\alpha,\beta) \iff N(\alpha) \in N(\beta)$. (When proving the relativised version of 3.6 there may be occurrences of an atomic formula $A(s_1,\ldots,s_n)$. These must be replaced by $R_A(s_1,\ldots,s_n)$ where R_A is the relation primitive recursive in A such that $R_A(\alpha_1,\ldots,\alpha_n) \iff A(N_A(\alpha_1),\ldots,N_A(\alpha_n))$.)

Clearly for admissible ordinals α

$$L_\alpha \models \theta \iff \alpha \models \theta_o$$

Now if θ is Π^n_m with $n > 0$ then θ_o is also Π^n_m and hence we can let θ_p be θ_o. If θ is Π^0_m with $m > 0$ then we must again be more careful. We can assume that θ is in prenex form with a sequence of quantifiers followed by a Π^0_o formula $\Psi(x_1,\ldots,x_k)$. Now Ψ determines a primitive recursive relation R and ordinals $\beta_1,\ldots,\beta_l$ such that for all α

$$L_\alpha \models \Psi(N(\alpha_1),\ldots,N(\alpha_k)) \iff \alpha \models R(\beta_1,\ldots,\beta_l,\alpha_1,\ldots,\alpha_k)$$

Now define θ_p by replacing $\Psi(x_1,\ldots,x_k)$ in θ by $R(\beta_1,\ldots,\beta_l,x_1,\ldots,x_k)$. Then θ_p is a Π^0_m sentenct of $\mathcal{L}_p$ satisfying the lemma.

We conclude this section with a characterisation of admissible ordinals that will be useful in the appendix. We state it in relativised form.

3.7. Theorem. Let A be a relation on ordinals. The ordinal β is admissible relative to $A \upharpoonright \beta$ if and only if for all $\alpha < \beta$ and all $R \subseteq {}^3ON$ that is primitive recursive in A if

$$\forall x < \beta \ \exists y < \beta \ R(\alpha,x,y)$$

then there is $\alpha < \lambda < \beta$ such that

$$\forall x < \lambda \ \exists y < \lambda \ R(\alpha,x,y) \ .$$

Proof. Note that this characterisation uses a restricted form of Π_2^0*-reflection. Hence it is only necessary to observe that this special form is sufficient for the proofs of 3.2. and 3.3.

§ 4. The relative sizes of the first order reflecting ordinals.

In this section we shall need some more results about ordinal recursion on an admissible ordinal. If f is a partial function on the admissible ordinal α then f is <u>α - partial recursive</u> if the graph of f is definable on L_α by a Σ_1^0 formula of $\mathcal{L}_\epsilon$.

As in theorem 4.4 of [8] we may prove:

4.1. <u>Normal Form theorem</u>. For each $n \geq 0$ there is a primitive recursive relation T_n and there is a primitive recursive function U such that if α is admissible and f is an n-ary α-partial recursive function then there is an $e < \alpha$ such that for $\alpha_1, \ldots, \alpha_n < \alpha$

$$f(\alpha_1, \ldots, \alpha_n) \simeq U(\mu_\alpha y\ T_n(e, \alpha_1, \ldots, \alpha_n, y))$$

Moreover e depends only on a Σ_1^0 formula of $\mathcal{L}_\epsilon$ that defines the graph of f on L_α . If this formula contains no constants then $e < \omega$. e is called an <u>α - index of f</u> .

Note the uniformity in this theorem. For example it follows that if $F : {}^n ON \to ON$ is primitive recursive then there is an $e < \omega$ such that $F \upharpoonright \alpha$ is α-recursive with α-index e for all admissible ordinals α .

Let us write $\{e\}_\alpha(\alpha_1, \ldots, \alpha_n)$ for $U(\mu_\alpha y\ T_n(e, \alpha_1, \ldots, \alpha_n, y))$. It will be useful to allow $n = 0$.

The next result is a uniform generalisation of Kleene's S - m - n theorem.

4.2. <u>Theorem</u>. For each $m > 0$ there is a primitive recursive function S_m such that for all admissible ordinals α if

$e, a_1, \ldots, a_m, \alpha_1, \ldots, \alpha_n < \alpha$ then $\{e\}_\alpha(a_1, \ldots, a_m, \alpha_1, \ldots, \alpha_n \simeq \{s_m(e, a_1, \ldots, a_m)\}_\alpha(\alpha_1, \ldots, \alpha_n)$.

This theorem may be proved roughly as follows: If f is an $m+n$-ary α-partial recursive function whose graph is defined by the Σ_1^0 formula $\theta(x_1, \ldots, x_n, x_{m+1}, \ldots, x_{m+n})$ on L_α then for $a_1 \ldots a_n < \alpha$ $\lambda\alpha_1 \ldots \alpha_n f(a_1, \ldots, a_m, \alpha_1, \ldots, \alpha_n)$ is also α-partial recursive, with graph defined by the Σ_1^0 formula $\theta(a_1, \ldots, a_m, x_1, \ldots, x_n)$ on L_α . Now S_m is chosen so that if e is the index of f determined by $\theta(x_1, \ldots, x_{m+n})$ then $S_m(e, a_1, \ldots, a_m)$ is the index of $\lambda\alpha_1, \ldots, \alpha_n f(a_1 \ldots a_m, \alpha_1 \ldots \alpha_n)$ determined by $\theta(a_1, \ldots, a_m), x_1, \ldots, x_n)$. We leave a detailed definition of S_m as a primitive recursive function independent of α to the imagination of the reader.

We now use theorem 4.1. to define universal Π_{m+1}^0 and Σ_{m+1}^0 formulae of $\mathcal{L}_p$. For each $n \geq 0$ let $\Sigma_1(x_0, \ldots, x_n)$ be $\exists y T_n(x_0, \ldots, x_n, y)$ and let $\Sigma_{m+1}(x_0, \ldots, x_n)$ be $\exists y \Pi_m(x_0, \ldots, x_n, y)$ for $m > 0$, where $\Pi_m(x_0, \ldots, x_k)$ is $\neg\Sigma_m(x_0, \ldots, x_k)$. Clearly $\Sigma_m(x_0, \ldots, x_n)$ is a Σ_m^0 formula of $\mathcal{L}_p$ and $\Pi_m(x_0, \ldots, x_n)$ is a Π_m^0 formula of $\mathcal{L}_p$ for each $m > 0$, $n \geq 0$.

Let us call two formulae of $\mathcal{L}_p$ $\theta_1(x_1, \ldots, x_n), \theta_2(x_1, \ldots, x_n)$ <u>equivalent on α</u> if for all $a_1, \ldots, a_n < \alpha$

$$\alpha \models \theta_1(a_1, \ldots, a_n) \iff \alpha \models \theta_2(a_1, \ldots, a_n) .$$

4.3. <u>Lemma</u>. Let $m > 0$. If $\varphi(x_1, \ldots, x_n)$ is a Σ_m^{0-} (Π_m^{0-}) formula of $\mathcal{L}_p$ then there is an $e < \omega$ such that $\varphi(x_1, \ldots, x_n)$ and $\Sigma_m(e, x_1, \ldots, x_n)(\Pi_m(e, x_1, \ldots, x_n))$ are equivalent on every admissible ordinal.

Proof. This is by induction on m. Note that the Π^0_m case follows from the Σ^0_m case by taking negations. If $m = 1$ and $\varphi(x_1,\ldots,x_n)$ is a Σ^{0-}_1 formula of $\mathcal{L}_p$, then, using the stability theorem 2.5 of [8], we may find a Σ^{0-}_1 formula $\varphi(x_1,\ldots,x_n,x_{n+1})$ of $\mathcal{L}_\in$ such that for admissible α and $\alpha_1,\ldots,\alpha_n$, $\beta < \alpha$

$$L_\alpha \models \varphi(\alpha_1,\ldots,\alpha_n,\beta) \iff \alpha \models \theta(\alpha_1,\ldots,\alpha_n) \;\&\; \beta = 0$$

But $\varphi(x_1,\ldots,x_{n+1})$ defines the graph of an α-partial recursive function on each admissible ordinal with index $e < \omega$ independent of α. Hence if α is admissible and $\alpha_1,\ldots,\alpha_n < \alpha$ then

$$\begin{aligned} \alpha \models \theta(\alpha_1,\ldots,\alpha_n) &\iff L_\alpha \models \varphi(\alpha_1,\ldots,\alpha_n,\beta) \text{ for some } \beta \\ &\iff \{e\}_\alpha(\alpha_1,\ldots,\alpha_n) \quad \text{is defined} \\ &\iff \alpha \models \exists y T_n(e,\alpha_1,\ldots,\alpha_n,y) \end{aligned}$$

Hence $\theta(x_1,\ldots,x_n)$ is equivalent to $\Sigma_1(e,x_1,\ldots,x_n)$ on admissibles.

Now suppose that the result has been proved for $m > 0$ and let $\varphi(x_1,\ldots,x_n)$ be Σ^0_{m+1}. Then we may assume that it has the form $\exists y_1\ldots\exists y_k\theta(y_1,\ldots,y_k,x_1,\ldots,x_n)$ for some Π^0_n formula $\theta(y_1,\ldots,y_k,x_1,\ldots,x_n)$.

Now let G be the graph of a primitive recursive function mapping k-tuples of ordinals one-one onto the ordinals. Then $\varphi(x_1,\ldots,x_n)$ is equivalent on every admissible to $\exists y\theta'(x_1,\ldots,x_n,y)$ where $\theta'(x_1,\ldots,x_n,y)$ is the Π^0_m formula

$$\forall y_1\ldots\forall y_k(G(y_1,\ldots,y_k,y) \to \theta(y_1,\ldots,y_k,x_1,\ldots,x_n)) .$$

By induction hypothesis there is an $e < \omega$ such that $\theta'(x_1,\ldots,x_n,y)$ is equivalent to $\Pi_m(e,x_1,\ldots,x_n,y)$ on every admissible. Hence $\varphi(x_1,\ldots,x_n)$ is equivalent to $\Sigma_m(e,x_1,\ldots,x_n)$ on every admissible.

4.4. <u>Corollary</u>. If $X \subseteq Ad$ then for $n > 0$

$$\alpha \in M_n(X) \iff \alpha \in X \;\&\; \forall a < \alpha[\alpha \models \Pi_n(a) \Rightarrow (\exists \beta \in X \cap \alpha)\, \beta \models \Pi_n(a)].$$

<u>Proof</u>. By theorem 1.10 $\alpha \in M_n(X)$ if and only if $\alpha \in X$ and for every Π_n^0 sentence φ of $\mathcal{L}_p$, $\alpha \models \varphi \Rightarrow (\exists \beta \in X \cap \alpha)\beta \models \varphi$. By lemma 4.3. and theorem 4.2. φ is a Π_n^0 sentence of $\mathcal{L}_p$ if and only if there is an ordinal a such that φ is equivalent to $\Pi_n(a)$ on every admissible. The corollary now follows when $X \subseteq Ad$.

Below we shall be conserned with operators F on classes of ordinals that have the following properties.

4.5. (i) $F(X) \subseteq L(X)$

(ii) $X \subseteq Y \Rightarrow F(X) \subseteq F(Y)$

(iii) $\lambda < \alpha \in F(X) \Rightarrow \alpha \in F(X \cap (\lambda,\alpha])$

where $(\lambda,\alpha] = \{\beta \mid \lambda < \beta \leq \alpha\}$.

It follows from (iii) that for all λ

$$F(X) \subseteq F(X \cap (\lambda,\infty]) \cup (\lambda+1)$$

where $(\lambda,\infty] = \{\beta \mid \lambda < \beta\}$.

Examples of such F are L, M, H_n, RM, M_n . Moreover, if F has these properties, then so does F^λ for $\lambda > 0$ and also F^Δ .

4.6. <u>Definition</u>. If F satisfies (i)-(iii) above and $n > 0$, then <u>F is Π_n^0-preserving</u> if there is a primitive recursive function $f: ON \to ON$ such that if $X = \{\alpha \in Ad \mid \alpha \models \Pi_n(a)\}$ then

a) $F(X) = \{\alpha \in Ad \mid \alpha \models \Pi_n(f(a))\}$

and b) $M_n(Ad) \subseteq X \cup \mu \Rightarrow M_n(Ad) \subseteq F(X) \cup \mu \in ON$ for all $\mu \in ON$.

4.7. Lemma. For $n > 0$, M_n is Π_n^0 preserving.

Proof. If $X = \{\alpha \in Ad \mid \alpha \models \Pi_n(a)\}$ then by 4.4. $\alpha \in M_n(X)$ if and only if $\alpha \in Ad \;\&\; \alpha \models [\Pi_{n+1}(a) \;\&\; \forall x \exists y(\Pi_n(x) \rightarrow R(a,x,y))]$ where R is the primitive recursive relation such that
$R(a,b,\beta) \iff \beta \models \Pi_n(b) \;\&\; \beta \in Ad \;\&\; \beta \models \Pi_{n+1}(a)$
So by 4.3. $M_n(X) = \{\alpha \in Ad \mid \alpha \models \Pi_{n+1}(e,a)\}$ for some $e < \omega$. Now if $f = \lambda x\, S_1(e,x)$ then f is primitive recursive and

$$M_n(X) = \{\alpha \in Ad \mid \alpha \models \Pi_{n+1}(f(a))\} .$$

Now let $M_{n+1}(Ad) \subseteq X \cup \mu$ and let $\alpha \in M_{n+1}(Ad)$. We must show that $\alpha \in M_n(X) \cup \mu$. If $\alpha < \mu$, then we are done. Otherwise $\alpha \in X$ so that $\alpha \in Ad$ and $\alpha \models \Pi_{n+1}(a)$. Now suppose that $\alpha \models \Pi_n(b)$. Then $\alpha \models \Pi_n(b) \wedge \Pi_{n+1}(a)$. As α is Π_{n+1}^0 reflecting on Ad there is a $\beta \in Ad \cap \alpha$ such that $\beta \models \Pi_n(b) \wedge \Pi_{n+1}(a)$. Hence $\beta \in X \cap \alpha$ and $\beta \models \Pi_n(b)$. Thus we have shown that $\alpha \in M_n(X)$.

4.8. Lemma. If F is Π_n^0-preserving, then so is F^Δ.

Proof. Let F be Π_n^0-preserving and let f be a primitive recursive function such that $F(\{\alpha \in Ad \mid \alpha \models \Pi_n(a)\}) = \{\alpha \in Ad \mid \alpha \models \Pi_n(f(a))\}$. Our first aim is to find a primitive recursive function g such that for admissible α and $a, c \in ON$

(1) $\alpha \models \Pi_n(g(a,c)) \iff c < \alpha \;\&\; (\forall b < c)\alpha \models \Pi_n(f(g(a,b))) \;\&\; \alpha \models \Pi_n(a)$

So let $\theta(x_1,x_2,x_3)$ be the formula

$$\Pi_n(x_2) \wedge \forall y \forall u \forall v(y < x_3 \wedge T_2(x_1,x_2,y,u) \wedge R(u,v) \rightarrow \Pi_n(v))$$

where $R = \{(u,v) \mid f(U(u)) = v\}$ is primitive recursive. Clearly this is Π_n^{o-} so that $\theta(x_1,x_2,x_3)$ is equivalent on admissibles to $\Pi_n(e_o,x_1,x_2,x_3)$ for some $e_o < \omega$. By a uniform version of the second recursion theorem on admissible ordinals there is an $e < \omega$ such that $\{e\}_\alpha(a,x) \simeq S_3(e_o,e,a,x)$ for $a,x < \alpha$ and admissible α. Now let $g = \lambda a,xS_3(e_o,e,a,x)$. Then on admissibles $\Pi_n(g(a,c))$ is equivalent to $\Pi_n(e_o,a,c)$ which is equivalent to $\theta(e,a,c)$. Hence for admissible α

$$\begin{aligned}
&\alpha \models \Pi_n(g(a,c))\\
\iff\ & c<\alpha \ \&\ \alpha\models\Pi_n(a) \ \&\ (\forall b<c)(\forall u,v<\alpha)(T_2(e,a,b,u) \ \&\ f(U(u)) = v \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Longrightarrow \alpha \models \Pi_n(v))\\
\iff\ & c<\alpha \ \&\ \alpha\models\Pi_n(a) \ \&\ (\forall b<c)\alpha\models\Pi_n(f(g(a,b)))\ .
\end{aligned}$$

so that (1) is proved.

Let $F^{(\beta)}(X) = \{\alpha > \beta \mid \alpha \in F^\beta(X)\}$.

Our next aim is to show that for all $\beta \in ON$

(2) $F^{(\beta)}(\{\alpha\in Ad)\alpha\models\Pi_n(a)\} = \{\alpha\in Ad \mid \alpha\models\Pi_n(g(a,\beta))\}$.

This will be proved by induction on β. Let $X = \{\alpha\in Ad|\alpha\models\Pi_n(a)\}$. By induction hypothesis, for $b < \beta < \alpha$

$$\begin{aligned}
\alpha \in F(F^b(X)) &\iff \alpha \in F(F^{(b)}(X)) \qquad \text{by 4.5. (iii)}\\
&\iff \alpha \in Ad \ \&\ \alpha\models\Pi_n(f(g(a,b)))\\
&\iff \alpha \in X \ \&\ \alpha\models\Pi_n(f(g(a,b)))\ .
\end{aligned}$$

Hence by (1)

$$\begin{aligned}
\alpha \in F^{(\beta)}(X) &\iff \beta<\alpha \ \&\ \alpha\in F^\beta(X)\\
&\iff \beta<\alpha \ \&\ (\forall b<\beta)\alpha\in F(F^\beta(X))\\
&\iff \beta<\alpha \ \&\ \alpha\in X \ \&\ (\forall b<\beta)\alpha\models\Pi_n(f(g(a,b)))\\
&\iff \beta<\alpha \ \&\ \alpha\in Ad \ \&\ \alpha\models\Pi_n(a) \ \&\\
&\qquad\qquad\qquad (\forall b<\beta)\alpha\models\Pi_n(f(g(a,b)))\\
&\iff \alpha\in Ad \ \&\ \alpha\models\Pi_n(g(a,\beta))\ .
\end{aligned}$$

So that (2) is proved.

Now we shall find a primitive recursive function f' such that

(3) $F^{\Delta}(\{\alpha \in Ad \mid \alpha \models \Pi_n(a)\}) = \{\alpha \in Ad \mid \alpha \models \Pi_n(f'(a))\}$

Let $X = \{\alpha \in Ad \mid \alpha \models \Pi_n(a)\}$. The formula $\forall x \forall y[g(z,x) = y \rightarrow \Pi_n(y)]$ is a Π_n^o- formula so that there is an $e_1 < \omega$ such that for admissible α and $a \in ON$

$$\begin{aligned}
\alpha \models \Pi_n(e_1, a) \quad &\Longleftrightarrow \quad \alpha \models \forall x \forall y[g(a,x) = y \rightarrow \Pi_n(y)] \\
&\Longleftrightarrow \quad (\forall b < \alpha)\alpha \models \Pi_n(g(a,b)) \\
&\Longleftrightarrow \quad (\forall b < \alpha)\alpha \in F^{(b)}(X) \qquad \text{by (2)} \\
&\Longleftrightarrow \quad (\forall b < \alpha)\alpha \in F^{b}(X) \\
&\Longleftrightarrow \quad \alpha \in F^{\Delta}(X)
\end{aligned}$$

But $F^{\Delta}(X) \subseteq X \subseteq Ad$ so that $F^{\Delta}(X) = \{\alpha \in Ad \mid \alpha \models \Pi_n(f'(a))$ where $f' = \lambda x S_1(e_1, x)$. So (3) is proved.

It now remains to show that if $X = \{\alpha \in Ad \mid \alpha \models \Pi_n(a)\}$ and $M_n(Ad) \subseteq X \cup \mu$ then $M_n(Ad) \subseteq F^{\Delta}(X) \cup \mu$. So let X, μ satisfy the above assumptions. We first show that for all $\beta \in ON$:

(4) $M_n(Ad) \subseteq F^{\beta}(X) \cup Max(\mu, \beta+1)$

This will be proved by induction on β. By induction hypothesis, if $b < \beta$, then

$$\begin{aligned}
M_n(Ad) &\subseteq F^{b}(X) \cup Max(\mu, b+1) \\
&\subseteq F^{(b)}(X) \cup Max(\mu, b+1)
\end{aligned}$$

But as F is Π_n^o-preserving, by (2), if $b < \beta$, then

$$\begin{aligned}
M_n(Ad) &\subseteq F(F^{(b)}(X)) \cup Max(\mu, b+1) \\
&\subseteq F(F^{b}(X)) \cup Max(\mu, \beta+1) \quad \text{by 4.5. (iii)}.
\end{aligned}$$

Hence

$$\begin{aligned}
M_n(Ad) &\subseteq (X \cap \bigcap_{b<\beta} F(F^{b}(X))) \cup Max(\mu, \beta+1) \\
&\subseteq F^{\beta}(X) \cup Max(\mu, \beta+1) .
\end{aligned}$$

Hence (4) is proved and now if $\alpha \in M_n(Ad)$ then if $\alpha < \mu$ we are done. Otherwise, by (4) $\alpha \in \bigcap_{\beta<\alpha} F^{\beta}(X)$ so that $\alpha \in F^{\Delta}(X) \cup \mu$. Thus $M_n(Ad) \subseteq F^{\Delta}(X) \cup \mu$.

We can now prove theorem 1.11.

If F iw Π^{o}_{n+1}-preserving, then $M_{n+1}(Ad) \subseteq F(Ad)$ as $Ad = \{\alpha \mid \alpha \models \Pi_n(e_o)\}$ for some $e_o < \omega$. Hence theorem 1.11. follows from the previous two lemmas.

4.9. Remark. If Y is a promitive recursive class of ordinals such that $Y \subseteq Ad$ and Ad is replaced by Y in definition 4.6., then the proofs of the previous two lemmas still hold so that we get that for $n > 0$:

$M_{n+1}(Y) \subseteq M^{\Delta}_n(Y)$, $(M^{\Delta}_n)^{\Delta}(Y)$, etc. ....

§ 5. Reflecting ordinals and indescribable cardinals.

In this section we will prove theorems 1.14 and 1.16.

5.1 Lemma. If k is 2-regular then $k>\omega$ and k is regular.

Proof. Let k be 2-regular. It suffices to show that every $g:k \to k$ has a witness. For a given g, let $F: {}^k k \to {}^k k$ be defined by $F(f)(\xi) = g(f(0))$ for all $f: k \to k$ and all $\xi < k$. F is clearly k-bounded. Let α be a witness for F. We show α is a witness for g. Let $\beta<k$ and $f: k \to k$ such that $f(\xi) = \beta$ for all $\xi<k$. Then $f''\alpha \subseteq \alpha$ and hence $F(f)''\alpha \subseteq \alpha$. Thus

$$g(\beta) = g(f(0) = F(f)(0) < \alpha .$$

Hence $g''\alpha \subseteq \alpha$.

5.2. Proof of Theorem 1.14: k is 2-regular iff k is strongly Π_1^1 - indescribable.

We show

$$\text{(a)}\ k \text{ 2-regular} \Rightarrow \begin{cases} \text{(b)}\ k \text{ is strongly inaccessible} \\ \quad \& \\ \text{(c)}\ k \text{ is } \Pi_1^1 \text{ - indescribable} \end{cases}$$

$$\Rightarrow \text{(d)}\ k \text{ is strongly } \Pi_1^1 \text{ - indescribable}$$

$$\Rightarrow \text{(a)} .$$

We first show (a) $\Rightarrow$ (b). Let k be 2-regular. Since k is regular it remains to show $\lambda < k \Rightarrow 2^\lambda < k$. Suppose not. Let $\lambda < k$ and $2^\lambda \geq k$. Let r map ${}^\lambda 2$ onto k. Define $F: {}^k k \to {}^k k$ by

$$F(f)(\xi) = \begin{cases} r(f\restriction\lambda) & \text{if } f''\lambda \subseteq 2 , \\ 0 & \text{otherwise,} \end{cases}$$

for $\xi < k$. F is k-bounded since $F(f)(\xi)$ is determined by values of f on $\lambda < k$. Let α be a witness for F . It is clear that $\alpha \geq 2$. Let $g: \lambda \to 2$ such that $r(g) > \alpha$ and let $f: k \to k$ so that $f \restriction \lambda = g$ and $f(\xi) = 0$ for $\xi \geq \lambda$. Then $f''\alpha \subseteq 2 \subseteq \alpha$. Since α is a witness for F , $F(f)(0) < \alpha$. But

$$F(f)(0) = r(f \restriction \lambda) = r(g) > \alpha$$

which is a contradiction.

To show (a) $\Rightarrow$ (c) let φ be a Π^1_1 sentence of $\mathcal{L}$ such that $k \models \varphi$. We must find an $0<\alpha<k$ such that $\alpha \models \varphi$. Let P be one of the standard bijective mappings of $On \times On$ onto On. and K, L be the associated pairing functions (cf. Lévy [11]) . We first switch from set quantifiers to quantifiers of binary relations which are characterictic functions, and then switch to the language with unary function quantifiers instead of set quantifiers. In this language with the aid of P, K, L we can put Π^1_1 formulas in a normal form (cf. Rogers [17], where this is done for formulas of second-order arithmetic). Thus there is a quantifier-free formula Q such that $k \models \forall f \exists \xi Q(f,\xi)$ and for every $\alpha \leq k$ which is closed under $\dot{P}$,

$$\alpha \models \varphi \Longleftrightarrow \alpha \models \forall f \exists \xi Q(f,\xi) .$$

Furthermore, Q can be chosen so that in Q there is no nesting of f(i.e. no terms of the form $f(f(\dots))$) . For a given f and ξ the truth or falsity of $Q(f,\xi)$ is determined by the values of f for finitely many arguments and the answers to finitely many questions about membership in the relations appearing in Q . Since there is no nesting of f

the finite set of arguments of f needed depends only on ξ and the relations in Q but not on f itself. Thus,

$$\forall\xi < k\, \exists\eta < k\ B(\xi,\eta) ,$$

where

$$B(\xi,\eta) \Longleftrightarrow \forall f \forall g [f{\restriction}\eta = g{\restriction}\eta \Rightarrow Q(f,\xi) \Longleftrightarrow Q(g,\xi)] .$$

Hence, since k is regular,

$$\forall\beta < k\, \exists\eta < k\ .\ C(\beta,\eta) ,$$

where

$$C(\beta,\eta) \Longleftrightarrow \forall\xi \leq \beta .\ B(\xi,\eta) ,$$

Let $h(\beta) =$ least $\eta\ .C(\beta,\eta)$. Then $h\colon k \to k$ and for all $f, g\colon k \to k$,

(2) $f{\restriction}h(\beta) = g{\restriction}h(\beta) \Rightarrow [\forall\xi \leq \beta .\ Q(f,\xi) \Longleftrightarrow Q(g,\xi)]$.

Let $G\colon {}^{k}k \to k$ so that

(3) $G(f) =$ least σ $[P''\sigma \times \sigma \subseteq \sigma\ \&\ \exists\xi \leq \sigma\ .\ Q(f,\xi)\ \&\ h(\xi) \leq \sigma]$,

and let $F\colon {}^{k}k \to {}^{k}k$ so that $F(f)(\beta) = G(f)$ for all $\beta < k$.

F is k-bounded since

$$\text{least } \xi .\ Q(f,\xi) = \text{least } \xi .\ Q(g,\xi) \Rightarrow F(f) = F(g).$$

Let α be a witness for F and let $f\colon \alpha \to \alpha$. We show $\exists\xi < \alpha .\ Q(f,\xi)$. Let $g\colon k \to k$ so that $g{\restriction}\alpha = f$. Then $g''\alpha \subseteq \alpha$ and $F(g)''\alpha \subseteq \alpha$ since α is a witness. Let $\delta =$ least $\xi .\ Q(g,\xi)$. Then $\delta, h(\delta) \leq G(g) < F(g)(0) < \alpha$ by definition of G .

Thus from (2) ,

$$\forall\xi \leq \delta\ \ Q(f,\xi) \Longleftrightarrow Q(g,\xi) ,$$

and hence least $\xi\ .Q(f,\xi) = \delta < \alpha$. Thus $\alpha \models \forall f \exists\xi\ Q(f,\xi)$

and since α is closed under P (by (3)), $\alpha \models \varphi$.

A proof that (b)&(c) $\Rightarrow$ (d) appears in Lévy [11] p. 217. It remains to show (d) $\Rightarrow$ (a). Let k be strongly Π^1_1 - indescribable and $F: {}^k k \to {}^k k$ be k-bounded. We show that F has a witness. Let

$$X = \{< f\restriction\gamma,\xi,\eta >: f\in {}^k k \ \&\ \xi,\eta,\gamma < k \ \&\ \forall g\in {}^k k[g\restriction\gamma = f\restriction\gamma \Rightarrow F(g)(\xi) = \eta]\}$$

Then $X \subseteq R(k)$. Note that $<f\restriction\gamma,\xi,\eta > \in X \Rightarrow F(f)(\xi) = \eta$. Since F is k-bounded,

$$R(k) \models \forall f \forall \xi \exists \gamma,\eta[<f\restriction\gamma,\xi,\eta > \in X] .$$

Since k is strongly Π^1_1 - indescribable there is an $\alpha < k$ such that

$$R(\alpha) \models \forall f \forall \xi \exists \gamma,\eta[< f\restriction\gamma,\xi,\eta > \in X] .$$

i.e. $0 < \alpha < k$ and

(4) $\forall f\in {}^\alpha\alpha\ \forall\xi < \alpha\exists\gamma,\eta < \alpha\ [< f\restriction\gamma,\xi,\eta > \in X \cap R(\alpha)]$.

We show α is a witness for F. Let $f: k \to k$ such that $f''\alpha \subseteq \alpha$. Since $f\restriction\alpha \in {}^\alpha\alpha$ and $f\restriction\alpha\restriction\gamma = f\restriction\gamma$ for $\gamma < \alpha$, we have from (4),

$$\forall\xi < \alpha\exists\eta < \alpha\ F(f)(\xi) = \eta \quad \text{, i.e. } F(f)''\alpha \subseteq \alpha .$$

5.3. Remark. In the proof of (d) $\Rightarrow$ (a), the assumption that F is k-bounded cannot be eliminated. For each k it is easy to define on $F: {}^k k \to {}^k k$ which is not k-bounded and has no witness.

5.4. Proof of Theorem 1.16: k is 2-admissible iff k is Π^0_3 - reflecting.

Suppose k is Π^0_3 - reflecting. Let $\{\xi\}_k$ map k - recurssive functions to k-recursive functions. We show ξ has a witness. By hypothesis,

$$\forall\beta < k\ [\{\beta\}_k\colon k \to k \Rightarrow \{\{\xi\}_k(\beta)\}_k\colon k \to k]\ .$$

By using the T predicate this is equivalent to

$$k \models \forall x[\forall y \exists z T(x,y,z) \to \forall y . T(\xi,x,y) \to \forall u \exists v T(U(y),u,v)].$$

Then sentence on the right is equivalent to a Π^0_3 sentence $\varphi(\xi)$.

Since k is Π^0_3 - reflecting (and hence Π^0_3 - reflecting on Ad) there is some $\alpha \in k \cap \mathrm{Ad}$ such that $\alpha = \varphi(\xi)$. But by the definition of $\varphi(\xi)$ this implies $\{\xi\}_\alpha$ maps α-recursive functions to α-recursive functions and hence α is a witness for ξ .

Now suppose k is 2-admissible and let φ be a Π^0_3 sentence of $\mathcal{L}p$ such that $k \models \varphi$. We show that k reflects φ . For simplicity we assume that φ is of the form $\forall x \exists y \forall z\ \psi(x,y,z)$ where ψ is a Σ^0_0 formula with constants less that k . Let α be admissible so that all constants in ψ are less than α . We introduce certain Gödel numbers of α-partial recursive functions which, by the uniformity of the Normal Form and S-m-n theorems, can be chosen to be independent of the particular choice of α . First choose $a < \alpha$ so that

$$\{S(a,\beta)\}\ (\gamma) \simeq \{a\}_\alpha(\beta,\gamma) \simeq \mu_\alpha\delta . \neg\psi(\beta,\gamma,\delta)\ .$$

Then,

$$\begin{aligned}
(5) \quad \alpha \models \varphi &\Leftrightarrow \forall\beta < \alpha \exists\gamma < \alpha\ \forall\delta < \alpha\ \psi(\beta,\gamma,\delta) \\
&\Leftrightarrow \forall\beta < \alpha\ \neg\ \forall\gamma < \alpha\delta < \alpha\ \neg\ \psi(\beta,\gamma,\delta) \\
&\Leftrightarrow \forall\beta < \alpha .\ \neg\ \{S(a,\beta)\}_\alpha\colon \alpha \to \alpha \\
&\Leftrightarrow \forall\beta < \alpha .\ \{\beta\}_\alpha\colon \alpha \to \alpha \Rightarrow \forall\gamma < \alpha \exists\delta < \alpha . S(a,\gamma) = \delta \,\&\, \delta \neq \beta\ .
\end{aligned}$$

Let g be a primitive recursive ordinal function such that

$$\{g(\beta)\}_\alpha(\gamma) \simeq \mu_\alpha\delta . S(a,\gamma) = \delta\ \&\ \delta \neq \beta$$

and let $\xi < \omega$ be a Gödel number (independent of α) of g . Then from (5) ,

$$(6) \quad \alpha \models \varphi \Leftrightarrow \forall\beta < \alpha . \{\beta\}_\alpha : \alpha \to \alpha \Rightarrow \{g(\beta)\}_\alpha : \alpha \to \alpha$$

$$\Leftrightarrow \forall\beta < \alpha . \{\beta\}_\alpha : \alpha \to \alpha \Rightarrow \{\{\xi\}_\alpha(\beta)\}_\alpha : \alpha \to \alpha$$

$$\Leftrightarrow \{\xi\}_\alpha \text{ maps } \alpha \text{ - recursive functions to}$$

α - recursive functions.

Since $k \models \varphi$, by (6) $\{\xi\}_k$ maps k - recursive functions to k - recursive functions. Since k is 2-admissible there is an $\alpha \in k \cap Ad$ which is a witness for ξ . But then by (6) , $\alpha \models \varphi$.

5.5. Remark. The definition of 2 - admissible given here is equivalent to the definition which appears in [2] . The full definition of n - admissible is given in [2] .

§ 6. Stability.

In this section we prove theorems 1.18 and 1.19.

Note that if $A \prec_{\Sigma_1^0} B$ and $A \subseteq C \subseteq B$ then $A \prec_{\Sigma_1^0} C$. It follows that if α is β-stable and $\alpha < \gamma < \beta$ then α is γ-stable. Hence the weakest stability property for an ordinal α is that of being $\alpha+1$-stable. 1.18 implies that even this weakest stability property determines ordinals with rather strong reflecting properties. But first we need:

6.1 Lemma

If α is $\alpha+1$-stable then α is admissible.

Proof. Let $L_\alpha \models \forall x \in a\, \varphi$ where φ is a Σ_1^0 formula $\mathcal{L}_\in$. Then $L_{\alpha+1} \models \forall x \in a\, \varphi^b$ where $b = L_\alpha \in L_{\alpha+1}$. Hence $L_{\alpha+1} \models \exists z\, \forall x \in a\, \varphi^z$. If α is $\alpha+1$-stable then $L_\alpha \models \exists z \forall x \in a\, \varphi^z$. Hence L_α satisfies Σ_1^0-collection. The lemma now follows, as α is clearly a limit ordinal so that L_α is rud closed.

Proof of 1.18: α is Π_0^1-reflecting if and only if α is $\alpha+1$-stable.

Let α be Π_0^1-reflecting. Let φ be a Σ_1^0 sentence of $\mathcal{L}_\in$, with constants only for sets in $\mathcal{L}_\in$, such that $L_{\alpha+1} \models \varphi$. We may assume that φ has the form $\exists x_1 \ldots \exists x_n \theta(x_1,\ldots,x_n)$ where $\theta(x_1,\ldots,x_n)$ is Π_0^0. So let $a_1,\ldots,a_n \in L_{\alpha+1}$ such that $L_{\alpha+1} \models \theta(a_1,\ldots,a_n)$. As $L_{\alpha+1} = \mathrm{Def}(L_\alpha)$ there are Π_0^1 formulae $\theta_1(v_0),\ldots,\theta_n(v_0)$ of $\mathcal{L}_\in$, with constants in L_α, such that $a_i = \{b \in L_\alpha \mid L_\alpha \models \theta_i(b)\}$ for $i = 1,\ldots,n$. Let θ' be obtained from $\theta(a_1,\ldots,a_n)$ by first replacing every occurence of $a_i \in s$ by $\exists y \in s\, \forall z(z \in y \leftrightarrow z \in a_i)$ and then replacing every occurence of $s \in a_i$ by $\theta_i(s)$ for $i = 1,\ldots,n$. Clearly θ' is a Π_0^1 sentence such that $L_\alpha \models \theta'$. As α is Π^1_0-reflecting there is a $\beta < \alpha$ such that $L_\beta \models \theta'$. Now if $a'_i = \{b \in L_\beta | L_\beta \models \theta_i(b)\}$ then $a'_i \in L_{\beta+1}$, as we may assume that x_1 actually occurs free in $\theta(x_1, \ldots, x_n)$ so that the constants of $\theta_i(v_o)$ are also constants of θ' and hence are constants for sets in L_β. It follows that $L_\alpha \models \theta(a'_1, \ldots, a'_n)$ and hence $L_\alpha \models \varphi$.

Conversely let α be $\alpha+1$-stable. Let σ_o be the Π^o_3-sentence given in theorem 2.4. Let $\Psi(x)$ be a Σ^o_1 formula of $\mathcal{L}_\in$ that defines L inside each admissible class A. Hence $\forall x\, \Psi(x)$ expresses $V = L$ and the transitive models of $\sigma_o \wedge \forall x \Psi(x)$ all have the form L_β for some admissible β. Now let φ be a Π^1_0 sentence of $\mathcal{L}_\in$ such that $L_\alpha \models \varphi$. Then $L_\alpha \models \varphi_1$ where φ_1 is $\varphi \wedge \sigma_o \wedge \forall x \Psi(x)$ is also a Π^1_0 sentence of $\mathcal{L}_\in$. Hence $L_{\alpha+1} \models \exists x(\mathrm{trans}(x) \wedge \varphi_1{}^{(x)})$ where $\varphi_1{}^{(x)}$ is obtained from φ_1 by restricting all quantifiers to x, and trans(x) is $\forall y \in x\ \forall z \in y(z \in x)$. As α is $\alpha+1$-stable it follows that $L_\alpha \models \exists x(\mathrm{trans}(x) \wedge \varphi_1{}^{(x)})$. Hence there is a transitive set in L_α that satisfies φ_1. But this must have the form L_β for some admissible β and $L_\beta \models \varphi$. It follows that α reflects φ, so that α is Π'_0-reflecting.

We now turn to the proof of 1.19. In fact we shall prove a generalisation of that result in 6.4. Some of the ideas in [4] will be basic to our proof. For a transitive set A let A^+ be the smallest admissible set such that $A \in A^+$. If $S \subseteq A$ we say that S is <u>$\Pi^n_m(\Sigma^n_m)$ over A</u> if $S = \{a \in A | A \models \varphi(a)\}$ for some $\Pi^n_m(\Sigma^n_m)$ formula $\varphi(x)$ of $\mathcal{L}_\in$. Theorem 3.1 (a) of [4] states that if A is a countable transitive set closed under unordered pairs then for $S \subseteq A$, S is Π^1_1 over A if and only if S is Σ^o_1 over A^+. The proof of this result in [4] may be made to yield the following formulation which gives us the extra information we shall need.

6.2 Theorem

(i) If $\varphi(v_1,\dots,v_n)$ is a Π^1_1 formula of $\mathcal{L}_\in$ then there is a Σ^0_1 formula $\varphi^+(v_0,v_1,\dots,v_n)$ of $\mathcal{L}_\in$ having the same constants as $\varphi(v_1,\dots,v_n)$ such that for every non-empty countable transitive set A and every admissible set B such that $A\in B$, if $a_1,\dots,a_n \in A$ then

$$A \models \varphi(a_1,\dots,a_n) \text{ iff } B \models \varphi^+(A,a_1,\dots,a_n).$$

(ii) If $\varphi(v_1,\dots,v_n)$ is a Σ^0_1 formula of $\mathcal{L}_\in$ then there is a Π^1_1 formula $\varphi^-(v_1,\dots,v_n)$ having the same constants as $\varphi(v_1,\dots,v_n)$ such that if A is an infinite transitive set containing the sets whose constants occur in $\varphi(v_1,\dots v_n)$ then for $a_1,\dots,a_n \in A$

$$A^+ \models \varphi(a_1,\dots,a_n) \text{ iff } A \models \varphi^-(a_1,\dots,a_n)$$

Proof. We shall require some familiarity with the infinitary languages $\mathcal{L}_B$ for admissible B. See for example [3].

(i) Let $\varphi(v_1,\dots,v_n)$ be a Π^1_1 formula of $\mathcal{L}_\in$. We may assume that it has the form $\forall X_1\dots\forall X_m\theta(v_1,\dots,v_n)$ where $\theta(v_1,\dots,v_n)$ is a Π^1_0 formula of $\mathcal{L}_\in$ with extra realtion symbols $X_1,\dots,X_m$. Given a non-empty transitive set A we may define the infinitary sentences $\Psi_0(A)$ and $\Psi_1(A)$ as follows: $\Psi_0(A)$ is $\bigwedge_{a\in A} \forall y(y\in a \leftrightarrow \bigvee_{b\in a}(y=b))$ and $\Psi_1(A)$ is $\forall x \bigvee_{a\in A}(a=x)$. Then the models of $\Psi_0(A)\wedge\Psi_1(A)$ are all isomorphic to $\langle A, \in\restriction A, a\rangle_{a\in A}$. Hence if $a_1,\dots,a_n \in A$ then

(1) $A \models \varphi(a_1,\dots,a_n)$ iff $\Psi_0(A)\wedge\Psi_1(A) \to \theta(a_1,\dots,a_n)$ is logically valid.

Note that if $A \in B$ where B is admissible then $(\Psi_0(A) \wedge \Psi_1(A) \rightarrow \theta(a_1,\ldots,a_n)) \in B$ i.e. it is a sentence of $\mathcal{L}_B$.

Now it is a routine matter, using [3] to find Σ_1^0 formulae of $\mathcal{L}_\in$ $\Psi(v_0,v_1,\ldots,v_{n+1})$ and $\chi(v_0)$, such that if $A,B,a_1,\ldots,a_n$ are as above then if $b \in B$

(2) $B \models \Psi(A,a_1,\ldots,a_n,b)$ iff $b = (\Psi_0(A) \wedge \Psi_1(A) \rightarrow \theta(a_1,\ldots,a_n))$, and if b is countable then

(3) $B \models \chi(b)$ iff b is a logically valid sentence of $\mathcal{L}_B$. (3) follows from the completeness theorem for countable infinitary sentences (see theorem 2.7 of [3]).

$\Psi(v_0,\ldots,v_{n+1})$ may be chosen to have the same constants as $\varphi(v_1,\ldots,v_n)$, while $\chi(v_0)$ may be chosen to have no constants.

Now let $\varphi^+(v_0,\ldots,v_n)$ be $\exists v_{n+1}(\Psi(v_0,\ldots,v_{n+1}) \wedge \chi(v_{n+1}))$. The result follows from (1)-(3) using the fact that $(\Psi_0(A) \wedge \Psi_1(A) \rightarrow \theta(a_1,\ldots,a_n))$ is countable if A is countable.

(ii) Let $\varphi(v_1,\ldots,v_n)$ be a Σ_1^0 formula of $\mathcal{L}_\in$. Let KP be the theory of admissible sets, as formulated in [3]. Then by 3.3 of [4], if A is a transitive set and $\mathfrak{B}$ is an end extension of $A \cup \{A\}$ that is a model of KP then $\mathfrak{B}$ is an end extension of A^+. Hence if $a_1,\ldots,a_n \in A$ then $A^+ \models \theta(a_1,\ldots,a_n)$ iff $\mathfrak{B} \models \theta(a_1,\ldots,a_n)$ for every A-model $\mathfrak{B}$ of KP where an <u>A-model of KP</u> is a model of KP that is an end-extension of $A \cup \{A\}$.

Now by the downward Lowenheim-Skolem theorem every A-model $\mathfrak{B}$ of KP has an elementary subsystem $\mathfrak{B}' \prec \mathfrak{B}$ that is an A-model of KP of the same cardinality as A, assuming that A is infinite. Every such A-model $\mathfrak{B}'$ is isomorphic to $\langle A,E\rangle$ for some $E \subseteq A \times A$. Then there is an $f : A \rightarrow A$ and $a \in A$ such that

(a) $\langle A,E\rangle$ is a model of KP

(b) $f: \langle A, \in\restriction A\rangle \cong \langle a_E, A\restriction a_E\rangle$ where $a_E = \{b \in A \mid bEa\}$

(c) $b_E \subseteq a_E$ for all $b \in a$.

It follows from the above that $A^+ \models \theta(a_1,\ldots,a_n)$ iff $\langle A,E\rangle \models \theta(f(a_1),\ldots,f(a_n))$ for all $E \subseteq A\times A$, $f: A \to A$ and $a \in A$ such that (a) & (b) & (c).

It is now a routine matter to find the required Π^1_1 formula obtained by formalizing the right hand side of the above equivalence.

6.3 Definition

An admissible set A is Π^1_1-reflecting if $A \models \varphi \Longrightarrow \exists a \in A$ $(a \models \varphi$ and a is transitive) for all Π^1_1 sentences φ of $\mathcal{L}_\in$.

The following is a generalisation of 1.19.

6.4 Theorem

The countable admissible set A is Π^1_1-reflecting if and only if $A \prec_{\Sigma^0_1} A^+$.

Proof. Let A be a countable admissible set that is Π^1_1-reflecting. Let φ be a Σ^0_1 sentence of $\mathcal{L}_\in$, with constants only for sets in A, such that $A^+ \models \varphi$. Let τ be $\forall x \exists y(x \in y)$. Then, by (6.2) (ii) with $n = 0$, $A \models \varphi^- \wedge \tau$. Hence $a \models \varphi^- \wedge \tau$ for some transitive $a \in A$. It follows that a is an infinite transitive set such that $a \models \varphi^-$. By (6.2) (ii) $a^+ \models \varphi$. But as $a^+ \subseteq A$ and φ is Σ^0_1 it follows that $A \models \varphi$. Hence $A \prec_{\Sigma^0_1} A^+$.

Conversely, let $A \prec_{\Sigma^0_1} A^+$ and let φ be a Π^1_1 sentence of $\mathcal{L}_\in$ such that $A \models \varphi$. Then by (6.2) (i) with $n = 0$, $A^+ \models \varphi^+(A)$. Hence $A^+ \models \varphi_1$ where φ_1 is the Σ^0_1 sentence $\exists x(\mathrm{trans}(x) \wedge \varphi^+(x))$. But φ_1 only has constants for sets in A, so that $A \models \varphi_1$ i.e. $A \models \varphi^+(a)$ for some transitive set $a \in A$. As A is countable so is a, so that by (6.2) (i) $a \models \varphi$. Thus A is Π^1_1-reflecting.

In order to obtain 1.19 we need:

(6.5) Lemma.

L_α is Π^1_1-reflecting iff α is Π^1_1-reflecting.

Proof. Let α be Π^1_1-reflecting. If φ is a Π^1_1 sentence of $\mathcal{L}_\in$ such that $L_\alpha \models \varphi$ then $L_\beta \models \varphi$ for some $\beta < \alpha$. But now $a = L_\beta$ is a transitive element of A such that $a \models \varphi$. Hence L_α is Π^1_1-reflecting.

Conversely, let L_α be Π^1_1-reflecting. Let σ be the Π^0_3 sentence $\sigma_0 \wedge \forall x \Psi(x)$ occurring in the proof of 1.12. If φ is a Π^1_1 sentence such that $L_\alpha \models \varphi$ then $L_\alpha \models \varphi \wedge \sigma$. Hence there is a transitive set $a \in L_\alpha$ such that $a \models \varphi \wedge \sigma$. But $a = L_\beta$ for some $\beta < \alpha$. So $L_\beta \models \varphi$ for some $\beta < \alpha$. Hence α is Π^1_1-reflecting.

Now 1.19 follows from 6.4 and 6.5 when we observe that $(L_\alpha)^+ = L_{\alpha^+}$ for every ordinal α.

§ 7. First order inductive definitions, I.

We begin by considering inductive definitions which are either recursive or closely related to recursive inductive definitions. These very simple cases illustrate some of the principles used in characterizing the closure ordinals of more complicated inductive definitions.

7.1. Definition. For any inductive definitions Γ_0, Γ_1 let

$$n \in [\Gamma_0, \Gamma_1](X) \iff n \in \Gamma_0(X) \vee [\Gamma_0(X) \subseteq X \,\&\, n \in \Gamma_1(X)]$$

Let $\Gamma = [\Gamma_0, \Gamma_1]$. In constructing the transfinite sequence $\langle \Gamma^\alpha : \alpha \in On \rangle$ one repeatedly applies Γ_0 until closure under Γ_0 is reached, (i.e. until a λ is reached such that $\Gamma_0(\Gamma^\lambda) \subseteq \Gamma^\lambda$) ; then Γ_1 is applied once; then Γ_0 is repeatedly applied until closure under Γ_0 is reached, etc. Γ_1 is applied only when closure under Γ_0 is reached. Note that if $\Gamma_0(\Gamma^\lambda) \subseteq \Gamma^\lambda$ then $\Gamma(\Gamma^\lambda) = \Gamma_1(\Gamma^\lambda)$. $|[\Gamma_0, \Gamma_1]|$ is the least λ such that both $\Gamma_0(\Gamma^\lambda) \subseteq \Gamma^\lambda$ and $\Gamma_1(\Gamma^\lambda) \subseteq \Gamma^\lambda$.

For any recursive relation R and inductive definition Γ, the truth or falsity of $R(n, \Gamma^\lambda)$ is determined by the answers to a finite number of questions about membership in Γ^λ. For limit λ, the answers to these questions are the same as the answers to the same questions about membership in Γ^ξ for suitably large $\xi < \lambda$. Hence for recursive R and limit λ,

$$(1) \quad R(n, \Gamma^\lambda) \Longrightarrow \exists \xi < \lambda \,.\, R(n, \Gamma^\xi) \,,$$

$$(2) \quad R(n, \Gamma^\lambda) \iff \exists \xi < \lambda\, \forall \delta \,.\, \xi \leq \delta < \lambda \Longrightarrow R(n, \Gamma^\delta)$$
$$\iff \forall \xi < \lambda\, \exists \delta \,.\, \xi \leq \delta < \lambda \,\&\, R(n, \Gamma^\delta) \,.$$

Using (1) and (2) we can prove the following trivial result.

Let $[\Pi^0_0,\Pi^0_0] = \{[\Gamma_0,\Gamma_1]: \Gamma_0,\Gamma_1 \in \Pi^0_0\}$, $[\Pi^0_0,\Pi^0_0,\Pi^0_0] = \{[\Gamma_0,[\Gamma_1,\Gamma_2]]: [\Gamma_0,\Gamma_1,\Gamma_2 \in \Pi^0_0\}$ etc...

7.2. <u>Proposition</u>. (i) $|\Pi^0_0| = \omega$,

(ii) $|[\Pi^0_0,\Pi^0_0]| = \omega^2$, $|[\Pi^0_0,\Pi^0_0,\Pi^0_0]| = \omega^3$, etc.

<u>Proof</u>. (i) Let $\Gamma \in \Pi^0_0$. Then for some recursive R , $n \in \Gamma(X) \Leftrightarrow R(n,X)$. Hence,

$$\begin{aligned} n \in \Gamma\Gamma^{\omega} &\Longrightarrow R(n,\Gamma^{\omega}) \\ &\Longrightarrow R(n,\Gamma^{\xi}) \quad \text{for some } \xi < \omega, \text{ by (1)} \\ &\Longrightarrow n \in \Gamma\Gamma^{\xi} \subseteq \Gamma^{\xi+1} \subseteq \Gamma^{\omega} . \end{aligned}$$

Thus $\Gamma\Gamma^{\omega} \subseteq \Gamma^{\omega}$ and hence $|\Gamma| \leq \omega$. To show $|\Pi^0_0| \geq \omega$, let $\Gamma_0(X) = \{0\} \cup \{\langle 1,x\rangle: x \in X\}$. Then $\Gamma_0 \in \Pi^0_0$. $0 \in \Gamma^{\infty}$ and $|0| = 0$; if $n \in \Gamma_0^{\infty}$ and $|n| = \xi$, then $\langle 1,n\rangle \in \Gamma_0^{\infty}$ and $|\langle 1,n\rangle| = \xi + 1$. Thus $|\Gamma_0| \geq \omega$.

(ii) Let $n \in \Gamma_0(X) \Leftrightarrow R_0(n,X)$ and $n \in \Gamma_1(X) \Leftrightarrow R_1(n,X)$ where R_0 and R_1 are recursive. Then as in the proof of (i) we have:

(3) <u>If limit</u> λ <u>then</u> $\Gamma_0\Gamma^{\lambda} \subseteq \Gamma^{\lambda}$ <u>and hence</u> $\Gamma\Gamma^{\lambda} = \Gamma_1\Gamma^{\lambda}$.

Now let $n \in \Gamma\Gamma^{\omega^2}$. We show $n \in \Gamma^{\omega^2}$. Since limit ω^2 , $n \in \Gamma_1\Gamma^{\omega^2}$ by (3), i.e. $R_1(n,\Gamma^{\omega^2})$. Then by (2) there is some $\xi < \omega^2$ such that $\forall\delta < \omega^2$. $\xi \leq \delta \Longrightarrow R_1(n,\Gamma^{\delta})$. Since the limit ordinals are cofinal in ω^2 there is some limit δ , $\xi \leq \delta < \omega^2$, and hence $R_1(n,\Gamma^{\delta})$. Thus $n \in \Gamma_1\Gamma^{\delta}$. But since limit δ, $\Gamma_1\Gamma^{\delta} = \Gamma\Gamma^{\delta}$ and hence $n \in \Gamma\Gamma^{\delta} \subseteq \Gamma^{\delta+1} \subseteq \Gamma^{\omega^2}$. Let Γ_0 be as above and let $\Gamma_1(X) = \{\langle 2,x\rangle: x \in X\}$. Let $\Gamma = [\Gamma_0,\Gamma_1]$. It is easy to show that $|\Gamma| \geq \omega^2$.

7.3. Remark: If R is Σ_1^0 then we still have (1) and the first equivalence in (2) so that 7.2 still holds if Σ_1^0 replaces Π_0^0 everywhere.

Note that $\Pi_0^0 \subseteq [\Pi_0^0,\Pi_0^0] \subseteq [\Pi_0^0,\Pi_0^0,\Pi_0^0] \subseteq \ldots \Pi_1^0$. Thus $\omega < \omega^2 < \omega^3 <\ldots< |\Pi_1^0|$. We have

7.4. Theorem. (Gandy) $|\Pi_1^0| = |\Sigma_2^0| = \omega_1$.

As $|\Sigma_2^0| \geq |\Pi_1^0| \geq |\Pi_1^0\text{-mon}| \geq \omega_1$ we have one half of the theorem. For the other half we will use the next two lemmas. These will also be used for getting upper bounds for other classes of first order inductive definitions.

7.5 Lemma. Let $\Gamma \in \Pi_0^1$. Then $\langle\Gamma^\xi : \xi<\lambda\rangle$ is uniformly Σ_1^0 on L_λ for $\lambda \in \mathrm{Ad}$. Hence for $\lambda \in \mathrm{Ad}$ Γ^λ is Σ_1^0 on L_λ.

Proof. If $\lambda \in \mathrm{Ad}$ and $x \subseteq \omega$ such that $x \in L_\lambda$ then $\Gamma(x)$ is Π_0^0 on L_λ as it is defined by a formula with quantifiers restricted to $\omega < \lambda$. Hence $\Gamma(x) \in L_\lambda$ as $\Gamma(x) \subseteq \omega$. So if $G(x,y) = \bigcup\{\Gamma(y^1 z \cap \omega) : z \in x\}$ then $G \restriction L_\lambda : L_\lambda \times L_\lambda \to L_\lambda$. Moreover $G\restriction L_\lambda$ is uniformly Σ_1^0 on L_λ for $\lambda \in \mathrm{Ad}$. Let $F(x) = G(x, F\restriction x)$. Then by 2.2 $F\restriction L_\lambda : L_\lambda \to L_\lambda$ and is uniformly Σ_1^0 on L_λ. By an easy induction we see that $\Gamma^\xi = F(\xi)$ for all $\xi \in \mathrm{ON}$, so that $\langle\Gamma^\xi : \xi<\lambda\rangle = F\restriction\lambda$ is uniformly Σ_1^0 on L_λ for $\lambda \in \mathrm{Ad}$. Hence Γ^λ is Σ_1^0 on L_λ as $x \in \Gamma \iff (\exists\xi<\lambda)\, x \in \Gamma^\xi$.

7.6. Lemma. Let $\langle\Gamma^\xi : \xi<\lambda\rangle$ be Σ_1^0 on L_λ where $\lambda \in \mathrm{Ad}$. Let R be recursive. Then

$$\forall x R(n,x,\Gamma^\lambda) \implies \exists\xi<\lambda \forall x R(n,x,\Gamma^\xi).$$

Proof. Suppose $\forall x R(n,x,\Gamma^\lambda)$ where $\lambda \in \mathrm{Ad}$. Then for each x, $\forall z < x R(n,z,\Gamma^\lambda)$. Since λ is a limit, by (2), $\forall z \leq x R(n,z,\Gamma^\xi)$

for some $\xi < \lambda$. Let $f(x) \simeq \mu\xi < \lambda\, \forall z < x\, R(n.z.\Gamma^{\xi})$. Then $f: \omega \to \lambda$ is λ-recursive. As $\omega < \lambda$ $\alpha = \sup_{n<\omega} f(n) < \lambda$. It remains to show that $\forall x R(n,x,\Gamma^{\alpha})$.

<u>Case</u> 1. limit α. Suppose that for some z_0, $\neg R(n,z_0,\Gamma^{\alpha})$. Then there is some $\xi < \alpha$ such that $\neg R(n,z_0,\Gamma^{\delta})$ whenever $\xi \leq \delta < \alpha$. Since limit α, there is some $x > z_0$ such that $\xi < f(x) < \alpha$ and hence $\neg R(n,z_0,\Gamma^{f(x)})$. But then by definition of f, $\forall z \leq x R(n,z,\Gamma^{f(x)})$. In particular $R(n,z_0,\Gamma^{f(x)})$ which is a contradiction.

<u>Case</u> 2. Not limit α. Then since f is non-decreasing there is some y such that for all $x \geq y$, $f(x) = \alpha$. But by definition of f this clearly implies $\forall x R(n,x,\Gamma^{\alpha})$.

We can now complete the proof of theorem 7.4. We must show:

7.7. <u>Lemma</u>. $|\Sigma_2^0| \leq \omega_1$.

<u>Proof</u>. Let Γ be Σ_2^0. Then

$$n \in \Gamma(X) \iff \exists y \forall x\, R(n,y,x,X)$$

for some recursive R. Then

$$\begin{aligned} n \in \Gamma(\Gamma^{\omega_1}) &\Longrightarrow \forall x\, R(n,y,x,\Gamma^{\omega_1}) \text{ for some } y < \omega, \\ &\Longrightarrow \forall x\, R(n,y,x,\Gamma^{\xi}) \text{ for some } \xi < \omega_1,\ y < \omega, \\ &\Longrightarrow n \in \Gamma(\Gamma^{\xi}) \subseteq \Gamma^{\omega_1}. \end{aligned}$$

Hence $\Gamma(\Gamma^{\omega_1}) \subseteq \Gamma^{\omega_1}$ so that $|\Gamma| \leq \omega_1$. A different proof appeared in [2]. The present proof is due to Grilliot.

As in the definitions of $[\Pi_0^0, \Pi_0^0]$, $[\Pi_0^0, \Pi_0^0, \Pi_0^0]$ etc. ... we may define $[\mathcal{C}_0, \mathcal{C}_1]$, $[\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2]$ etc. ... for any classes $\mathcal{C}_0, \mathcal{C}_1, \ldots$ of i.d.'s.

7.8. Lemma.

(i) $|[\Pi_1^0, \Pi_n^0]| \leq |[\Sigma_2^0, \Sigma_{n+1}^0]| \leq$ least element of $M_{n+1}(Ad)$;

(ii) $|[\Pi_1^0, \Pi_n^0, \Pi_m^0]| \leq |[\Sigma_2^0, \Sigma_{n+1}^0, \Sigma_{m+1}^0]| \leq$ least element of $M_{m+1}(M_{n+1}(Ad))$ etc. ...

Proof. The first inequalities in (i) and (ii) are trivial, so we turn the last inequalities.

(i) Let $\Gamma = [\Gamma_0, \Gamma_0]$ where $\Gamma_0 \in \Sigma_2^0$ and $\Gamma_1 \in \Sigma_{n+1}^0$. Then $\Gamma \in \Pi_0^1$ and hence $\langle \Gamma^\xi : \xi < \lambda \rangle$ is Σ_1^0 on L_λ for $\lambda \in Ad$, so that by the proof of lemma 7.7

(4) If $\lambda \in Ad$ then $\Gamma_0(\Gamma^\lambda) \subseteq \Gamma^\lambda$ and hence $\Gamma(\Gamma^\lambda) = \Gamma_1(\Gamma^\lambda)$. Suppose first that n is even. Then for some recursive R

$$a \in \Gamma_1(X) \iff \exists x_0 \, \forall x_1 \ldots \exists x_n R(a, x_0, \ldots, x_n, X)$$

Hence by (2) and (4), if $\lambda \in Ad$ then

$$\begin{aligned} a \in \Gamma(\Gamma^\lambda) &\iff a \in \Gamma_1(\Gamma^\lambda) \quad \text{[illegible]}) \\ &\iff (\exists x_0 \in \omega)(\forall x_1 \in \omega)\ldots(\exists x_n \in \omega) R(a, x_0, \ldots, x_n, \Gamma^\lambda) \\ (5) \qquad &\iff (\exists x_0 \in \omega)(\forall x_1 \in \omega)\ldots(\exists x_n \in \omega)(\exists \xi < \lambda)(\forall \delta < \lambda)[\xi \leq \delta \Rightarrow \\ &\qquad\qquad R(a, x_0, \ldots, x_n, \Gamma^\delta)] \\ &\iff L_\lambda \models Q(a) \end{aligned}$$

for some Σ_{n+2}^0 formula $Q(v)$ of $\mathcal{L}_\epsilon$ that is independent of $\lambda \in Ad$.

Now let κ be the least element of $M_{n+1}(Ad)$. Suppose $a \in \Gamma(\Gamma^\kappa)$. Then as $\kappa \in Ad$ it follows from (5) that $L_\kappa \models Q(a)$. As κ is Σ_{n+2}^0-reflecting on Ad there is a $\lambda < \kappa$ such that $\lambda \in Ad$ and $L_\lambda \models Q(a)$. Hence by (5) $a \in \Gamma(\Gamma^\lambda) \subseteq \Gamma^\kappa$. Thus $\Gamma(\Gamma^\kappa) \subseteq \Gamma^\kappa$ and hence $|\Gamma| \leq \kappa$.

If n is odd then for some recursive R

$$a \in \Gamma_1(X) \iff \exists x_0 \forall x_1 \ldots \forall x_n R(a, x_0, \ldots, x_n, X) .$$

Hence using (2) and (4) again, if $\lambda \in \mathrm{Ad}$ then

$$a \in \Gamma(\Gamma^\lambda) \iff (\exists x_0 \in \omega) \ldots (\forall x_n \in \omega)(\forall \xi < \lambda)(\exists \delta < \lambda)[\xi \leq \delta \,\&\, R(a, x_0, \ldots, x_n, \Gamma^\delta)]$$

$$\iff L_\lambda \models Q(a)$$

for some Σ^0_{n+2} formula $Q(v)$ of $\mathcal{L}_\epsilon$ that is independent of $\lambda \in \mathrm{Ad}$. The rest of the proof is as before.

(ii) This follows the same pattern as the proof of (i). Let $\Gamma = [\Gamma_0, \Gamma_1, \Gamma_2]$ where $\Gamma_0 \in \Sigma^0_2$, $\Gamma_1 \in \Sigma^0_{n+1}$ and $\Gamma_2 \in \Sigma^0_{m+1}$. The proof of (i) shows that

(4') If $\lambda \in M_{n+1}(\mathrm{Ad})$ then $[\Gamma_0, \Gamma_1](\Gamma^\lambda) \subseteq \Gamma^\lambda$ and hence $\Gamma(\Gamma^\lambda) = \Gamma_2(\Gamma^\lambda)$.

Then as in (5) if $\lambda \in M_{n+1}(\mathrm{Ad})$

(5') $a \in \Gamma(\Gamma^\lambda) \iff L_\lambda \models Q(a)$

for some Σ^0_{m+2} formula $Q(v)$ of $\mathcal{L}_\epsilon$ that is independent of $\lambda \in M_{n+1}(\mathrm{Ad})$. The rest of the proof is as in (i).

In the next section we will prove results which will enable us to reverse the inequalities in this lemma.

§ 8. Closed classes of inductive definitions.

In this section we formulate the notion of a closed class $\mathscr{C}$. The results in this section will enable us to give characterisations of $|\mathscr{C}|$ and $\mathrm{Ind}(\mathscr{C})$ for many of these classes.

8.1. Definition. $f: \Delta \leq_m \Gamma$ if

(a) f is a recursive function and $\{f(e)\}$ is total for all e ;

(b) if $\{e\}: X \leq_m Y$ then $\{f(e)\}: \Delta(X) \leq_m \Gamma(Y)$.

$\Delta \leq_m \Gamma$ if $f: \Delta \leq_m \Gamma$ for some f. $\Delta \leq_1 \Gamma$ is defined similarly.

8.2. Theorem. If $\Delta \leq_m \Gamma$ then $\Delta^{\infty} \leq_m \Gamma^{\infty}$ and $|\Delta| \leq |\Gamma|$. Similarly, with $\leq_m$ replaced by $\leq_1$.

This is an immediate consequence of the following:

8.3. Lemma. If $\Delta \leq_m \Gamma$ there is a recursive function g such that for all α, $g: \Delta^{\alpha} \leq_m \Gamma^{\alpha}$.

Proof. Let $f: \Delta \leq_m \Gamma$. By the recursion theorem there is an e such that $\{e\} = \{f(e)\} = g$, say. g is total since $\{f(e)\}$ is. We show by induction on α that $g: \Delta^{\alpha} \leq_m \Gamma^{\alpha}$. Suppose

$$\{e\} = g : \Delta^{\beta} \leq_m \Gamma^{\beta} \quad \text{and hence} \quad g = \{f(e)\}: \Delta\Delta^{\beta} \leq_m \Gamma\Gamma^{\beta}$$

for all $\beta < \alpha$. Then,

$$\begin{aligned} x \in \Delta^{\alpha} &\iff \exists \beta < \alpha.\ x \in \Delta\Delta^{\beta} \\ &\iff \exists \beta < \alpha.\ g(x) \in \Gamma\Gamma^{\beta} \\ &\iff g(x) \in \Gamma^{\alpha}. \end{aligned}$$

8.4. Definition. Γ is $\mathscr{C}$-complete if $\Gamma \in \mathscr{C}$ and $\mathscr{C} = \{\Delta: \Delta \leq_m \Gamma\}$.

8.5. Theorem. If Γ is $\mathscr{C}$-complete then $|\mathscr{C}| = |\Gamma|$ and $\mathrm{Ind}(\mathscr{C}) = \{X : X \leq_m \Gamma^{\infty}\}$.

Proof. Let Γ be $\mathscr{C}$-complete. As $\Gamma \in \mathscr{C}$, $|\mathscr{C}| \geq |\Gamma|$ and $\mathrm{Ind}(\mathscr{C}) \supseteq \{X : X \leq_m \Gamma^{\infty}\}$. If $\Delta \in \mathscr{C}$ then $\Delta \leq_m \Gamma$ and hence by 2.2. $|\Delta| \leq |\Gamma|$ and $\Delta^{\infty} \leq_m \Gamma^{\infty}$. Hence $|\mathscr{C}| \leq \Gamma$ and $\mathrm{Ind}(\mathscr{C}) \subseteq \{X : X \leq_m \Gamma^{\infty}\}$.

8.6. Theorem. There is a Π^n_{m+1}-complete operator. Similarily for Σ^n_{m+1}.

Proof. We shall need the following folklore result, which is well-known when $n = 0$ or $n = 1$, but is equally true for larger n.

8.7. Proposition. There is a universal Π^n_{m+1} operator. Similarily for Σ^n_{m+1}.

By this we mean a Π^n_{m+1} operator Γ such that every Π^n_{m+1} operator Δ has the form $\Delta(X) = \Gamma_a(X) = \{x \mid \langle a,x\rangle \in \Gamma(X)\}$ for some $a \in \omega$.

We will show that Γ is Π^n_{m+1}-complete.
Let $\Delta_1(X) = \{\langle e,x\rangle \mid x \in \Delta(\{e\}^{-1}X)\}$.
When $n > 0$, Δ_1 is easily seen to be Π^n_{m+1} and hence has the form Γ_a for some $a \in \omega$. Now let f be a recursive function such that $\{f(e)\}(x) = \langle a,\langle e,x\rangle\rangle$. Then $f\colon \Delta \leq_m \Gamma$.

When $n = 0$ we must be more careful as Δ_1 may not be Π^0_{m+1}. We will define a Π^0_{m+1} operator Δ_2 such that if $\{e\}$ is total then $\langle e,x\rangle \in \Delta_1(X) \Longleftrightarrow \langle e,x\rangle \in \Delta_2(X)$. Then $f\colon \Delta \leq_m \Gamma$ if we let $\Delta_2 = \Gamma_a$.

Let $\varphi(X,x)$ be a Π^0_{m+1} formula defining Γ. By separating out positive and negative occurrences of X in $\varphi(X,x)$ we may write the formula as $\theta(X,\omega-X,x)$ where $\theta(X,Y,x)$ contains only positive occurrences of X and Y. Then

$$\langle e,x\rangle \in \Delta_1(X) \iff \theta(\{e\}^{-1}X,\omega-\{e\}^{-1}X,x)$$

Now if m is odd let

$$\Delta_2(X) = \{\langle e,x\rangle \mid \theta(\{e\}^{-1}X,\{e\}^{-1}(\omega-X),x)\}$$

and if m is even let

$$\Delta_2(X) = \{\langle e,x\rangle \mid \theta(\omega-\{e\}^{-1}(\omega-X),\omega-\{e\}^{-1}X,x)\}.$$

Then in each case Δ_2 is Π^0_{m+1}.

8.8. Definition.

$\mathscr{C}$ is closed if

(a) There is a $\mathscr{C}$-complete operator ;

(b) $\Gamma_1,\Gamma_2 \in \mathscr{C} \implies \Gamma_1 \cup \Gamma_2, \Gamma_1 \cap \Gamma_2 \in \mathscr{C}$

(c) Every recursive operator is in $\mathscr{C}$

The following result is now trivial.

8.9. Theorem. Π^n_{m+1} and Σ^n_{m+1} are closed.

In order to obtain characterisations of $|\mathscr{C}|$ and $\mathrm{Ind}(\mathscr{C})$ for closed classes $\mathscr{C}$ we will need a method for constructing notation systems $\mathfrak{M}=(M,|\,|)$ which is more general than that mentioned in the introduction. We shall first give an example which bears some resemblance to Kleene's systems of notations for the constructive ordinals. We define a transfinite sequence of sets $\langle M_\xi : \xi \in ON\rangle$. In the definition $|a| = \mu\xi(a \in M_{\xi+1})$. $\lambda x[b](x,X)$ is the b'th function primitive recursive in X in a recursive enumeration, uniform in X , of the functions primitive recursive in X .

$$M_0 = \emptyset$$

$$M_{\alpha+1} = M_\alpha \cup \{0\} \cup \{\langle 1,a,b\rangle : a \in M_\alpha \;\&\; \forall x[b](x, M_{|a|}) \in M_\alpha\}.$$

$$M_\lambda = \bigcup_{\xi<\lambda} M_\xi \quad \text{for limit } \lambda.$$

$$M = \bigcup_{\xi \in ON} M_\xi$$

Note that the definition of M has the appearance of a set inductively defined by an i.d. But the situation is complicated by the fact that the definition of $M_{\alpha+1}$ depends not only on the previously defined M_α, but also on $\langle M_{|a|} : a \in M_\alpha\rangle$. Given any sequence $\langle M_\xi : \xi \in ON\rangle$ we use the following notation.

$$M = \bigcup\{M_\xi : \xi \in ON\}$$

$$|a| = \mu\xi(a \in M_{\xi+1}) \quad \text{for } a \in M$$

$$|M| = \operatorname{Sup}\{|x| : x \in M\}$$

$$M^*_\alpha = \{\langle x,y\rangle : x,y \in M_\alpha \;\&\; |x| \leq |y|\} \quad \text{for } \alpha \in ON$$

$$M^* = \bigcup\{M^*_\alpha : \alpha \in ON\}$$

If $X \subseteq \omega$ let $\mathcal{F}(X) = \{x : \langle x,x\rangle \in X\}$ and

$$X_{<x} = \{y : \langle y,x\rangle \in X \;\&\; \langle x,y\rangle \notin X\}.$$

Then clearly $M_\alpha = \mathcal{F}(M^*_\alpha)$ and $M_{|a|} = (M^*_\alpha)_{<a}$ for $a \in M_\alpha$.

Hence the definition of $M_{\alpha+1}$ above may be written

$$M_{\alpha+1} = M_\alpha \cup \Theta(M^*_\alpha) \quad \text{where}$$

$$\Theta(X) = \{0\} \cup \{\langle 1,a,b\rangle : a \in \mathcal{F}(X) \;\&\; \forall x[b](x, X_{<a}) \in \mathcal{F}(X)\}.$$

Notice that Θ is Π^0_1. We will see below that M^* is inductively defined by a Π^0_1 i.d. We now generalize the above procedure to an arbitrary Θ.

8.10. Definition. For any i.d. Θ,

$$\mathfrak{M}^{\Theta} = (M^{\Theta}, |\,|) \text{ is defined by:}$$

$$M_0 = \emptyset$$

$$M_{\alpha+1} = M_\alpha \cup \Theta(M_\alpha^*)$$

$$M_\lambda = \bigcup\{M_\alpha : \alpha < \lambda\} \text{ if limit } \lambda$$

$$M^{\Theta} = \bigcup\{M_\alpha : \alpha \in ON\}.$$

Although M^{Θ} does not have an obvious inductive definition we show that M^* does.

8.11. Definition. For any i.d. Θ, $\Theta_{\leq}$ is defined by $\Theta_{\leq}(X) =$

$$\{\langle x,y\rangle : x \in \mathcal{J}(X) \;\&\; y \in \Theta(X)\setminus\mathcal{J}(X)\} \cup \{\langle x,y\rangle : x, y \in \Theta(X)\setminus\mathcal{J}(X)\}.$$

8.12. Remark. For closed $\mathcal{C}$ note that $\Theta \in \mathcal{C} \Longrightarrow \Theta_{\leq} \in \mathcal{C}$.

8.13. Lemma. For all α $\Theta_{\leq}^{\alpha} = M_\alpha^*$ and hence $|\Theta_{\leq}| = |M^{\Theta}|$ and $\Theta_{\leq}^{\infty} = M^*$.

Proof. Note that $M_{\alpha+1}^* = M_\alpha^* \cup \Theta_{\leq}(M_\alpha^*)$. The result now follows by induction on α.

8.14. Equivalence Theorem.

If $\mathcal{C}$ is closed and Θ is $\mathcal{C}$-complete then $|\mathcal{C}| = |M^{\Theta}|$ and $\mathrm{Ind}(\mathcal{C}) = \{X \subseteq \omega : X \leq_m M^{\Theta}\}$.

Proof. If $\mathcal{C}$ is closed and $\Theta \in \mathcal{C}$ it follows from 8.12 and 8.13 that $|\mathcal{C}| \geq |M^{\Theta}|$ and $\mathrm{Ind}(\mathcal{C}) \supseteq \{X \subseteq \omega : X \leq_m M^{\Theta}\}$ as $\Theta_{\leq} \in \mathcal{C}$ and $x \in M^{\Theta} \Longleftrightarrow \langle x,x\rangle \in \Theta_{\leq}^{\infty}$.

For the converse it is sufficient to prove the following.

8.15. Lemma. If $\Gamma \leq_m \Theta$ then there is a recursive function f such that for all α $f : \Gamma^{\alpha} \leq_m M_\alpha$; hence $\Gamma^{\infty} \leq_m M^{\Theta}$ and $|\Gamma| \leq |M^{\Theta}|$.

Proof. Let f_0 be a recursive function such that $\{f_0(e)\}(x) \simeq \langle\{e\}(x),\{e\}(x)\rangle$. So $\{e\}^{-1}\mathcal{F} = \{f_0(e)\}^{-1}$. Let $f_1 : \Gamma \leq_m \Theta$. Then for total $\{a\}$, $\Gamma\{a\}^{-1} = \{f_1(a)\}^{-1}\Theta$. Then

$$\Gamma\{e\}^{-1}\mathcal{F} = \Gamma\{f_0(e)\}^{-1} = \{f_1 f_0(e)\}^{-1}\Theta .$$

Now choose e such that $\{e\} = \{f_1 f_0(e)\}$ and let $f = \{e\}$.

Then $\Gamma f^{-1}\mathcal{F} = f^{-1}\Theta$. Hence $\Gamma f^{-1}(M_\alpha) = \Gamma f^{-1}\mathcal{F}(M_\alpha^*) = f^{-1}\Theta(M_\alpha^*)$, so that $f^{-1}M_\alpha \cup \Gamma(f^{-1}M_\alpha) = f^{-1}M_\alpha \cup f^{-1}\Theta(M_\alpha^*) = f^{-1}(M_\alpha \cup \Theta(M_\alpha^*)) = f^{-1}M_{\alpha+1}$. It follows by induction on α that $\Gamma^\alpha = f^{-1}M_\alpha$; i.e. $f: \Gamma^\alpha \leq_m M_\alpha$.

In § 7 we have seen how to prove, for certain $\mathcal{C}$, that $|\mathcal{C}| \leq \kappa$ where κ is the least reflecting ordinal of a certain kind. In order to show that $|\mathcal{C}| = \kappa$ we will choose a 'good' notation system $\mathcal{M} = \langle M, |\,|\rangle$ such that $|M| \leq |\mathcal{C}|$ and show that M has the required reflection property. As M is a set of notations for the ordinals $<|M|$, statements about ordinals $<|M|$ can be rewritten as statements about M. The reflection property for $|M|$ will then follow from closure properties of M. The coding lemma below gives a formulation of this rewriting process for Σ_1^0 statements.

8.16. Definition. A notation system $\mathcal{M} = \langle M, |\,|\rangle$ is good if $\mathcal{M} = \mathcal{M}^\Theta$ where $\Theta(X) = \Xi(X) \cup \Phi(X)$ and

$\Xi(X) = \{o\} \cup \{\langle 1,a,b\rangle : a \in \mathcal{H}(X)\ \&\ \forall x[b](x, X_{<a}) \in \mathcal{H}(X)\}$
$\cup \{\langle 2,a,b\rangle : a \in \mathcal{J}(X)$ or $b \in \mathcal{J}(X)\}$, and $\Phi(X)$ is always disjoint from $\{o\} \cup \{\langle 1,a,b\rangle : a,b \in \omega\} \cup \{\langle 2,a,b\rangle : a,b \in \omega\}$.

If $\mathcal{M}$ is good then an ordinal $\lambda \leq |M|$ is $\mathcal{M}$-good if $\Xi(M_\lambda^*) \subseteq M_\lambda$. Thus $|M|$ is $\mathcal{M}$-good, but usually there will be $\mathcal{M}$-good ordinals $<|M|$.

8.17. Coding Lemma.

Let $\mathcal{M} = \langle M, |\,| \rangle$ be a good notation system and let $T_{\mathcal{M}} = \{(x,\alpha) : \alpha \in On \,\&\, x \in M_\alpha\}$. Then

(i) Every $\mathcal{M}$-good ordinal is in $Ad(T_{\mathcal{M}})$.

(ii) For every Σ_1^{0-} formula $\varphi(v_1,\dots,v_n)$ of $\mathcal{L}_p(T_{\mathcal{M}})$ there is a primitive recursive function h such that for every $\mathcal{M}$-good ordinal λ

$$a_1,\dots,a_n \in M_\lambda \,\&\, \lambda \models \varphi(|a_1|,\dots,|a_n|) \iff h(a_1,\dots,a_n) \in M_\lambda .$$

(iii) If λ is $\mathcal{M}$-good then for $X \subseteq \omega$

$$X \text{ is } \lambda\text{-r.e. in } T_{\mathcal{M}} \restriction \lambda \iff X \leq_m M_\lambda .$$

This lemma will be proved in the appendix.

8.18. Corollary. Let $\mathcal{M}$, $T_{\mathcal{M}}$ be as in the lemma, and let $f(\alpha) = \mu n[n \in M \,\&\, |n| = \alpha]$ for $\alpha < |M|$. Then for each $\mathcal{M}$-good ordinal λ $f \restriction \lambda : \lambda \to \omega$ is a λ-recursive in $T_{\mathcal{M}} \restriction \lambda$ injection.

Proof. $f(\alpha) = \mu n[(n,\alpha+1) \in T_{\mathcal{M}} \,\&\, (n,\alpha) \notin T_{\mathcal{M}}]$. Hence $f \restriction \lambda$ is λ-recursive in $T_{\mathcal{M}} \restriction \lambda$ for $\mathcal{M}$-good λ. It is clearly an injection.

8.19. Theorem. Let $\mathcal{C} \supseteq \Pi_1^0$ be closed and let Γ be $\mathcal{C}$-complete. Let $\lambda = |\mathcal{C}|$ and $A_\Gamma = \{(n,\alpha) : \alpha < \lambda \,\&\, n \in \Gamma^\alpha\}$. Then

(i) λ is admissible relative to A_Γ ;

(ii) λ is projectible to ω relative to A_Γ ;

(iii) $Ind(\mathcal{C}) = \{X \subseteq \omega : X \text{ is } \lambda\text{-r.e. in } A_\Gamma\}$.

Proof. Let $\Theta(X) = \Xi(X) \cup \Phi(X)$ where $\Phi(X) = \{\langle 3,a \rangle : a \in \Gamma(X)\}$. Then as $\Xi \in \Pi_1^0 \subseteq \mathcal{C}$ and $\mathcal{C}$ is closed it follows that $\Theta \in \mathcal{C}$.

Also $\lambda x\langle 3,x\rangle : \Gamma(X) \leq_m \Theta(X)$ for all X and hence $f' : \Gamma \leq_m \Theta$ where $f : \Gamma \leq_m \Gamma$ and f' is a recursive function such that $\{f'(e)\}(x) \simeq \langle 3, \{f(e)\}(x)\rangle$. Hence Θ is $\mathcal{C}$-complete. Now $\mathcal{M} = \mathcal{M}^{\Theta}$ is a good notation system and by the equivalence theorem $|\mathcal{C}| = |M|$ and $\mathrm{Ind}(\mathcal{C}) = \{X \subseteq \omega : X \leq_m M\}$. Hence by the coding lemma and its corollary the theorem follows as long as we replace A_Γ by $T_{\mathcal{M}} \restriction \lambda$. It only remains to show that A_Γ and $T_{\mathcal{M}} \restriction \lambda$ are λ-recursive in each other. But by 8.15 there is a recursive function h such that $h : \Gamma^\alpha \leq_m M_\alpha$ for all α. Hence $(n,\alpha) \in A_\Gamma \iff (h(n),\alpha) \in T_{\mathcal{M}} \restriction \lambda$, so that A_Γ is λ-recursive in $T_{\mathcal{M}} \restriction \lambda$. For the converse note that as Γ is $\mathcal{C}$-complete and $\Theta_{\leq} \in \mathcal{C}$ it follows that $g : \Theta_{\leq} \leq_m \Gamma$ for some g. Hence $g : M^*_\alpha \leq_m \Gamma^\alpha$ for all α, by 8.3 and 8.13. So

$$(n,\alpha) \in T_{\mathcal{M}} \restriction \lambda \iff (g(\langle n,n\rangle,\alpha) \in A_\Gamma ,$$

showing that $T_{\mathcal{M}} \restriction \lambda$ is λ-recursive in A_Γ.

§ 9. First order inductive definitions, II.

We are now ready to characterise the ordinals of first order inductive definitions.

9.1. Theorem.

(i) $|\Pi_1^0|$ is the least element of Ad;

(ii) $|[\Pi_1^0, \Pi_n^0]|$ is the least element of $M_{n+1}(\mathrm{Ad})$;

(iii) $|[\Pi_1^0, \Pi_m^0, \Pi_n^0]|$ is the least element of $M_{n+1}(M_{m+1}(\mathrm{Ad}))$.

etc.....

9.2 Remark. By 7.8 this theorem is also true when each Π_k^0 is replaced by Σ_{k+1}^0.

Before proving the theorem we derive some immediate consequences.

9.3 Corollary. $|\Pi_n^0| = |\Sigma_{n+1}^0| = \pi_{n+1}^0 = \sigma_{n+2}^0$.

Proof. $|\Pi_0^0| = |\Sigma_1^0| = \omega = \pi_1^0$ by 1.9(i), 7.2 and 7.3.
$|\Pi_1^0| = |\Sigma_2^0| = \omega_1 = \pi_2^0$ by 1.8 and 7.4. For $n > 1$ $[\Pi_1^0, \Pi_n^0] = \Pi_n^0$ and by 1.9 (iv) $M_{n+1}(\mathrm{Ad}) = M_{n+1}(\mathrm{ON})$. Hence $|\Pi_n^0| = |\Sigma_{n+1}^0| = \pi_{n+1}^0$ by 9.1 and 9.2.
By 1.9(iii) $\pi_n^0 = \sigma_{n+1}^0$ for all n.

9.4 Corollary.

(i) $|[\Pi_1^0, \Pi_0^0]|$ is the least recursively inaccessible ordinal; $|[\Pi_1^0, \Pi_0^0, \Pi_0^0]|$ is the least recursively inaccessible limit of recursively inaccessible ordinals, etc....

(ii) $|[\Pi_1^0, \Pi_1^0]|$ is the least recursively Mahlo ordinal; $|[\Pi_1^0, \Pi_1^0, \Pi_1^0]|$ is the least recursively hyper-Mahlo ordinal, etc. ...

We now turn to the proof of the theorem. By 7.8 it only remanins to prove:

9.5 Lemma.

(i) $\alpha \in Ad$ for some $\alpha \leq |\Pi_1^0|$;

(ii) $\alpha \in M_{n+1}(Ad)$ for some $\alpha \leq |[\Pi_1^0, \Pi_n^0]|$;

(iii) $\alpha \in M_{r+1}(M_{m+1}(Ad))$ for some $\alpha \leq |[\Pi_1^0, \Pi_m^0, \Pi_n^0]|$; etc. ...

Proof.

(i) This follows from theorem 7.4, whose proof assumed the result $|\Pi_1^0\text{-mon}| \geq \omega_1$. To give a direct proof let $\mathcal{M} = \mathcal{M}^{\Theta}$ where $\Theta = \Xi$, and let $\alpha = |M|$. Then $\mathcal{M}$ is a good notation system so that $\alpha \in Ad$, by the coding lemma. As $\Theta \in \Pi_1^0$, so is $\Theta_{\leq}$ so that $\alpha = |\Theta_{\leq}| \leq |\Pi_1^0|$.

(ii) First assume that n is odd.
Let $\mathcal{M} = \mathcal{M}^{\Theta}$ where $\alpha \in \Theta(X) \iff \alpha \in \Xi(X) \vee [\Xi(X) \subseteq J(X) \& \alpha \in \Phi_n^1(X)]$, $\Phi_n^1(X) = \{\langle 3, e\rangle : e \in \Phi_n(\mathcal{F}(X))\}$ and

$a \in \Phi_n(X) \iff (\forall x_1 \in X)(\exists x_2 \in X) \ldots (\forall x_n \in X)\ [a](x_1, \ldots, x_n) \in X$.
Here $\lambda x_1, \ldots, x_n\ [a](x_1, \ldots, x_n)$ is the a'th n-ary primitive recursive function in a recursive enumeration of them.

An easy argument shows that
$\Theta_{\leq} = [\Xi_{\leq}, (\Phi_n^1)_{\leq}]$, so that $\Theta_{\leq} \in [\Pi_1^0, \Pi_n^0]$ as $(\Phi_n^1)_{\leq} \in \Pi_n^0$. $\mathcal{M}$ is a good notation system so that by the coding lemma $\alpha = |M| \in Ad$.
Let φ be a Π_{n+1}^0 sentence of $\mathcal{L}_p$ such

that $\alpha \models \varphi$. We may assume that φ has the form

$$\forall x_1 \exists x_2 \dots x_n \ \Psi(x_1, \dots, x_n, |c_1|, \dots, |c_k|)$$

where Ψ is a Σ_1^{0-} formula of $\mathcal{L}_p$ and $c_1, \dots, c_k \in M$. By the coding lemma there is a primitive recursive function h such that for all $\mathcal{M}$-good ordinals λ

$$a_1, \dots, a_{n+k} \in M_\lambda \ \& \ \lambda \models \Psi(|a_1|, \dots, |a_{n+k}|) \Longleftrightarrow h(a_1, \dots, a_{n+k}) \in M_\lambda$$

Now choose $e \in \omega$ such that

$$[e](a_1, \dots, a_n) = h(a_1, \dots, a_n, c_1, \dots, c_k).$$

Then it follows that for $\mathcal{M}$-good λ

$$\lambda \models \varphi \Longleftrightarrow e \in \Phi_n(M_\lambda)$$
$$\Longleftrightarrow \ < 3, e > \ \in \Phi_n^1(M_\lambda^*) \subseteq M_{\lambda+1}$$

Hence as $\alpha \models \varphi$ and α is $\mathcal{M}$-good

$$< 3, e > \ \in M$$

Now if $\lambda = |<3,e>|$ then $\lambda < \alpha$, λ is $\mathcal{M}$-good, and hence admissible, and $<3,e> \in \Phi_n^1(M_\lambda^*)$ so that $\lambda \models \varphi$. Thus $\alpha \in M_{n+1}(Ad)$ and $\alpha = |\Theta_{\leq}| \leq |[\Pi_1^0, \Pi_n^0]|$, as required.

When $n > 0$ is even the proof is as above except that
$$a \in \Phi_n(X) \Longleftrightarrow (\forall x_1 \in X)(\exists x_2 \in X) \dots (\exists x_n \in X)[a](x_1, \dots, x_n) \notin X,$$
and the Π_{n+1}^0 sentence φ now has the form

$\forall x_1 \exists x_2 \dots \exists x_n \ \Psi(x_1, \dots, x_n, |c_1|, \dots, |c_k|)$ with $\Psi, c_1, \dots, c_k$ as before.

In case $n = 0$ let $\Phi_n(X) = X$. Then as before $\alpha = |M| \leq |[\Pi_1^0, \Pi_n^0]|$ and $\alpha \in Ad$. In order to show that $\alpha \in M_1(Ad)$ we must show that α is a limit of admissibles. So let $\beta < \alpha$. Then $\beta = |a|$ for some $a \in M$. Then $<3,a> \in M$ as $\Xi(M^*) \subseteq M$. Let $\lambda = |<3,a>| < \alpha$. Then λ is $\mathcal{M}$-good and hence admissible, and $\beta = |a| < \lambda$. So $\alpha \in M_1(Ad)$.

(iii) Let $\mathfrak{M} = \mathfrak{M}^{\Theta}$ where

$$a \in \Theta(X) \Longleftrightarrow a \in \Xi(X) \vee [\Xi(X) \subseteq \mathcal{F}(X) \ \& \ a \in \Phi_m^1(X)]$$
$$\vee [\Xi(X) \cup \Phi_m^1(X) \subseteq \mathcal{F}(X) \ \& \ a \in \Phi_n^{11}(X)]$$

where $\Phi_n^{11}(X) = \{<5,e> : e \in \Phi_n(J(X))\}$

and Φ_m, Φ_m^1 are as in (ii). Then as in (ii) $\Theta_{\leq} = [\Xi_{\leq}, (\Phi_m^1)_{\leq}, (\Phi_n^{11})_{\leq}] \in [\Pi_1^0, \Pi_m^0, \Pi_n^0]$ so that $\alpha = |M| = |\Theta_{\leq}| \leq |[\Pi_1^0, \Pi_m^0, \Pi_n^0]|$.

As in the proof of (ii) we may show that $\alpha \in M_{m+1}(Ad)$. Moreover we may show that $|<5,e>| \in M_{m+1}(Ad)$ whenever $<5,e> \in M$. Hence using once more the argument in the proof of (ii) we can show that $\alpha \in M_{n+1}(M_{m+1}(Ad))$.

The next result characterises $\mathrm{Ind}(\mathcal{C})$ for certain classes of first order inductive definitions.

<u>9.6 Theorem.</u>

If $\mathcal{C}$ is any of the classes $\Pi_1^0, [\Pi_1^0, \Pi_n^0], [\Pi_1^0, \Pi_m^0, \Pi_n^0]$, etc... and $\lambda = |\mathcal{C}|$ then there is a $\Gamma \in \mathcal{C}$ such that $\lambda = |\Gamma|$ and

$$\mathrm{Ind}(\mathcal{C}) = \{X \subseteq \omega : X \leq_m \Gamma^{\infty}\}$$
$$= \{X \subseteq \omega : X \text{ is } \lambda\text{-r.e.}\}$$

<u>9.7 Remark.</u>

This result also applies to the classes Π_{n+1}^0 and to the classes obtained from the ones considered by replacing each Π_k^0 by Σ_{k+1}^0.

<u>Proof</u>. The proof has the same form in each case. We illustrate with $\mathcal{C} = [\Pi_1^0, \Pi_n^0]$. In the proof of 9.5 (ii) a good notation system $\mathfrak{M} = \mathfrak{M}^{\Theta}$ is defined such that $|M| \leq \lambda$ and $|M| \in M_{n+1}(Ad)$. But λ is the first element of $M_{n+1}(Ad)$ by 9.1. Hence $|M| = \lambda$. So if $\Gamma = \Theta_{\leq} \in \mathcal{C}$ then $\lambda = |\Gamma|$. By the coding lemma $\{X \subseteq \omega : X \text{ is } \lambda\text{-r.e.}\} \subseteq \{X \subseteq \omega : X \leq_m M\} \subseteq \{X \subseteq \omega : X \leq_m \Gamma^{\infty}\}$, as

$M \leq_m M^* = \Gamma^\infty$. Hence $\{X \subseteq \omega$: X is λ-r.e.$\} \subseteq \mathrm{Ind}(\mathcal{C})$ as $\Gamma \in \mathcal{C}$. $\Lambda \in \mathcal{C}$ implies that $\Lambda^\infty = \mathcal{A}^\lambda$ is λ - r.e. by 7.5. Hence $\mathrm{Ind}(\mathcal{C}) \subseteq \{X \subseteq \omega$: X is λ - r.e.$\}$, proving the theorem.

For completeness we conclude this section with the following easily proved result.

9.8 Theorem.

(i) $\mathrm{Ind}(\Pi^0_0) = \{X \subseteq \omega$: X is r.e.$\}$.

(ii) $\mathrm{Ind}(\Sigma^0_1) = \{X \subseteq \omega$: X is a "recursive" union of arithmetical sets$\}$.

§ 10. Higher order inductive definitions, I.

In the previous section it was shown that the closure ordinals of certain classes of first-order inductive definitions are reflecting ordinals of a prescribed form. In this section we obtain corresponding results for higher type inductive definitions. The techniques are similar to those of § 7 and § 9.

In Lemma 7.5. it was shown that for $\Gamma \in \Pi^1_0$, $\langle \Gamma^\xi : \xi < \lambda \rangle$ is uniformly Σ^0_1 on L_λ for $\lambda \in Ad$. For other Γ this need not be the case and this makes the characterization of $|\Pi^m_n|$ for $m,n > 0$ somewhat more difficult.

In the following lemma the class A is some relation on ordinals.

10.1. Lemma. Suppose $m,n > 0$ and $\Gamma \in \Pi^m_n$. Let X be a class of limit ordinals greater than ω, and $\kappa \in X$ be $\Pi^m_n(A)$-reflecting on X. If $< \Gamma^\xi : \xi < \lambda >$ is uniformly Σ^1_0 on $L_\lambda[A]$ for $\lambda \in X$, then $|\Gamma| \leq \kappa$. Similarly with $\Pi^m_n(A)$ and Π^m_n replaced by $\Sigma^m_n(A)$ and Σ^m_n, respectively.

Proof. Let $\Gamma \in \Pi^m_n$. Then for some Π^m_n formula $\varphi(y,Y)$ of $\mathcal{L}_\in$, for all $Y \subseteq \omega$

$$n \in \Gamma(Y) \iff \omega \models \varphi(n,Y) .$$

Let $\psi_0(z)$ be the formula $z \in \omega$, and $\psi_{k+1}(Z^{k+1})$ be $\forall Y^k[Z^{k+1}(Y^k) \rightarrow \psi_k(Y^k)]$. Then each ψ_{k+1} is a Π^k_1 formula (in the constant ω). Let $\varphi^*(n,X)$ be obtained from φ by restricting each quantifier of type k to ψ_k. Then $\varphi^* \in \Pi^m_n$ and for $\lambda > \omega$ and $Y \subseteq \omega$,

$$n \in \Gamma(Y) \iff L_\lambda[A] \models \varphi^*(n,Y) .$$

Let $\varphi^*(y,Y)$ be $\vec{Q}\vec{Z}.\psi(\vec{Z},y,Y)$ where $\vec{Q}\vec{Z}$ is a sequence of quantified variables of appropriate type for a prenex Π^m_n formula and ψ is Σ^0_0. Then for $\lambda \in X$,

$$(1) \quad s \in \Gamma(\Gamma^\lambda) \Leftrightarrow L_\lambda[A] \models \vec{Q}\vec{Z}.\psi(\vec{Z},s,\Gamma^\lambda)$$
$$\Leftrightarrow L_\lambda[A] \models \vec{Q}\vec{Z}\forall\xi \in On\, \exists\delta \in On[\xi \leq \delta \wedge \psi(\vec{Z},s,\Gamma^\delta)]$$
$$\Leftrightarrow L_\lambda[A] \models \vec{Q}\vec{Z}\forall\xi \in On\, \exists\delta \in On\, \exists y[\xi \leq \delta \wedge R(\delta,y) \wedge \psi(\vec{Z},s,y)]$$
$$\Leftrightarrow L_\lambda[A] \models \varphi_1(s),$$

where R is a Σ^1_0 formula of $\mathcal{L}_\in(A)$ (independent of $\lambda \in X$) such that for $\delta < \lambda$, $y \in L_\lambda[A]$

$$\Gamma^\delta = y \Leftrightarrow L_\lambda[A] \models R(\delta,y),$$

and $\varphi_1(s)$ is the sentence appearing immediately above it in (1). Clearly φ_1 is a Π^m_n formula of $\mathcal{L}_\in(A)$. Now let $\kappa \in X$ be $\Pi^m_n(A)$ - reflecting and $s \in \Gamma\Gamma^\kappa$. We show $s \in \Gamma^\kappa$. $L_\kappa[A] \models \varphi_1(s)$ by (1). Since φ_1 is a Π^m_n formula of $\mathcal{L}_\in(A)$ there is some $\lambda \in X \cap \kappa$ such that $L_\lambda[A] \models \varphi_1(s)$. Hence by (1), $s \in \Gamma\Gamma^\lambda \subseteq \Gamma^\kappa$.

10.2. <u>Definition</u>. For a given i.d. Γ let

$$A_\Gamma(x,y) \Leftrightarrow y \in On \;\&\; x \in \Gamma^y.$$

Let $\pi^m_n(\Gamma)$ be the least $\Pi^m_n(A_\Gamma)$-reflecting ordinal; similarly for $\sigma^m_n(\Gamma)$.

10.3. <u>Theorem</u>. Let $m,n > 0$ and Γ be complete Π^n_m. Then $|\Pi^m_n| = \pi^m_n(\Gamma)$. Similarly $|\Sigma^m_n| = \sigma^m_n(\Gamma)$ if Γ is complete Σ^m_n.

<u>Proof</u>. We prove this for Π^m_n. The proof for Σ^m_n is similar. Let Γ be complete Π^m_n. For $\lambda \in Ad(A_\Gamma)$

and $\xi < \lambda$, $\Gamma^\xi = \{x \in \omega: A_\Gamma(x,\xi)\} \in L_\lambda[A_\Gamma]$,
by Σ^0_0-separation. And since for $y \in L_\lambda[A_\Gamma]$,

$$\Gamma^\xi = y \iff L_\lambda[A_\Gamma] \models [\forall x \in \omega\ A_\Gamma(x,\xi) \to x \in y] \wedge \forall x \in y . A_\Gamma(x,\xi) ,$$

it follows that $\langle \Gamma^\xi : \xi < \lambda \rangle$ is uniformly Σ^0_1 on $L_\lambda[A_\Gamma]$ for $\lambda \in Ad(A_\Gamma)$. Letting $X = Ad(A_\Gamma)$ and $\varkappa = \pi^m_n(\Gamma)$ in Lemma 1, we see that $|\Pi^m_n| \leq \pi^m_n(\Gamma)$.

To show $|\Pi^m_n| \geq \pi^m_n(\Gamma)$ it suffices to show $|\Pi^m_n|$ is $\Pi^m_n(\Gamma)$-reflecting. Let Θ be as in the proof of 8.19 and let $\mathfrak{M} = \langle M, \| \rangle = \mathfrak{M}^\Theta$. By 8.14 $|M| = |\Pi^m_n|$.
If $T_{\mathfrak{M}}$ and h are as in the proof of 8.19 then

$$A_\Gamma(x, \alpha) \iff T_{\mathfrak{M}}(h(x),\alpha)$$
$$\iff \exists_y \in \omega[h(x) = y \ \&\ T_{\mathfrak{M}}(y,\alpha)] .$$

Since h is recursive, the predicate $h(x) = y$ is Σ^0_1 on L_ω and hence Σ^0_0 on $L_\lambda[T_{\mathfrak{M}}]$ for $\lambda > \omega$. Hence A_Γ is Σ^0_0 on $L_\lambda[T_{\mathfrak{M}}]$ for $\lambda > \omega$. So if φ is a Π^m_n sentence of $\mathcal{L}_\epsilon(A_\Gamma)$ there is a Π^m_n sentence φ^* of $\mathcal{L}_\epsilon(T\ )$ such that for $\lambda > \omega$ $L_\lambda[A_\Gamma] \models \varphi \iff L_\lambda[T_{\mathfrak{M}}] \models \varphi^*$.
Hence it suffices to show that $|M|$ is $\Pi^m_n(T_{\mathfrak{M}})$-reflecting. This will follow from the next lemma. Let $\varphi(v_1,\ldots,v_l)$ be a Π^{m-}_nformula of $\mathcal{L}_p(T_{\mathfrak{M}})$ with the indicated free variables.

10.4. <u>Lemma</u>. There is a Π^m_n i.d. Ψ , such that for $\mathfrak{M}$-good λ and $c_1,\ldots,c_l \in M_\lambda$

$$\lambda \models \varphi(|c_1|,\ldots,|c_l|) \iff \langle c_1,\ldots,c_l \rangle \in \Psi(M^*_\lambda) .$$

<u>Proof.</u> This will be in five parts. Assume throughout that λ ranges over $\mathfrak{M}$-good ordinals.

(1) If R is primitive recursive in $T_{\mathfrak{M}}$ then by the coding lemma there is a primitive recursive function h_R , independent of λ , such that for $a_1,\ldots,a_n \in M$

$$\lambda \models R(|a_1|,\ldots,|a_k|) \iff h_R(a_1,\ldots,a_k) \in M_\lambda .$$

In particular

$$a,b \in M_\lambda \ \& \ |a| = |b| \iff h_{=}(a,b) \in M_\lambda \ .$$

(2) Let $\mathcal{G}(Y,X)$ if and only if $X \subseteq \omega$ and $Y \subseteq \omega x \omega$ is the graph of a bijection $f:\omega \cong Q \subseteq X$ such that

(i) $x,y \in Q \ \& \ h_{=}(x,y) \in X \Rightarrow x = y$, and

(ii) $y \in X \Rightarrow \exists x \in Q \ h_{=}(x,y) \in X$.

It should be clear that $\mathcal{G}$ is arithmetical. Moreover $\mathcal{G}(Y,M_\lambda)$ holds if and only if Y is the graph of a bijection $f: \omega \cong Q \subseteq M_\lambda$ such that $Y^* = \lambda x|f(x)|$ is a bijection: $\omega \cong \lambda$. Hence $\exists Y \mathcal{G}(Y,M_\lambda)$.

(3) If R is primitive recursive in T_m let $\theta_R(Y,X,x_1,\ldots,x_k)$ be the Σ_1^{0-}formula of $\mathcal{L}_p$

$$\exists y_1 \ldots \exists_{y_k} \exists_y [\bigwedge_{1 \leq i \leq k} Y(x_i,y_i) \wedge X(y) \wedge (h_R(y_1,\ldots,y_k) = y)]$$

Then if $\mathcal{G}(Y,M_\lambda)$ and $a_1,\ldots,a_k \in \omega$

$$\lambda \models R(Y^*(a_1),\ldots,Y^*(a_k)) \iff \omega \models \theta_R(Y,M_\lambda,a_1,\ldots,a_k)$$

(4) Let $\varphi^*(Y,X,v_1,\ldots,v_l)$ be obtained from $\varphi(v_1,\ldots,v_l)$ by replacing every atomic formula $R(x_1,\ldots,x_k)$ by $\theta_R(Y,X,x_1,\ldots,x_k)$. Then φ^* is a Π_n^{m-}formula of $\mathcal{L}_p$, and if $\mathcal{G}(Y,M_\lambda)$ and $a_1,\ldots,a_l \in \omega$ then

$$\lambda \models \varphi(Y^*(a_1),\ldots,Y^*(a_l)) \iff \omega \models \varphi^*(Y,M_\lambda,a_1,\ldots,a_l) \ .$$

(5) $\Psi(X)$ may now be defined to be the set of $\langle c_1,\ldots,c_l \rangle$ such that $c_1,\ldots,c_l \in \mathcal{F}(X)$ and for all Y such that $\mathcal{G}(Y,\mathcal{F}(X))$ and all $a_1,\ldots,a_l$, $b_1,\ldots,b_l$, if $\bigwedge_{1 \leq i \leq l} (Y(a_i,b_i) \wedge h_{=}(b_i,c_i) \in \mathcal{F}(X))$ then

$$\omega \models \varphi^*(Y,\mathcal{F}(X),a_1,\ldots,a_l) \ .$$

Then Ψ is a Π_n^m i.d. that satisfies the lemma.

We can now complete the proof of the theorem. Let $|M| \models \varphi$ where φ is a Π_n^m sentence of $\mathcal{L}_p(T_m)$. We must find $\lambda < |M|$ such that $\lambda \models \varphi$. φ must have the form

$\varphi(|a_1|,\dots,|a_l|)$ for some Π^{m}_{n}-formula of $\mathcal{L}_p(T_{\mathfrak{m}})$ and $a_1,\dots,a_l \in M$. Let Ψ be the i.d. given by lemma 10.4. Then as Ψ is Π^m_n and Γ is complete Π^m_n there is a g: $\Psi(X) \leq_m \Gamma(X)$ for all X. Let $a = g(\langle a_1,\dots,a_l\rangle)$. Then for $\mathfrak{m}$-good λ

$$\begin{aligned}\lambda \models \varphi &\Longleftrightarrow \langle a_1,\dots,a_l\rangle \in \Psi(M^*_\lambda)\\ &\Longleftrightarrow a \in \Gamma(M^*_\lambda)\\ &\Longleftrightarrow \langle 3,a\rangle \in \Theta(M^*_\lambda)\end{aligned}$$

But $|M| \models \varphi$. Hence $\langle 3,a\rangle \in \Theta(M^*) \subseteq M$. Let $\lambda = |\langle 3,a\rangle|$. Then $\lambda < |M|$ is $\mathfrak{m}$-good and $\langle 3,a\rangle \in \Theta(M^*_\lambda)$. Hence $\lambda \models \varphi$.

<u>10.5. Corollary</u>. For m, n > 0 $|\Pi^m_n|$ is Π^m_n-reflecting and $|\Sigma^m_n|$ is Σ^m_n-reflecting. Hence $|\Pi^m_n| \geq \pi^m_n$ and $|\Sigma^m_n| \geq \sigma^m_n$.

<u>Proof</u>. By theorem 10.3 $|\Pi^m_n|$ is $\Pi^m_n(\Gamma)$-reflecting and hence Π^m_n-reflecting. Similarily for Σ^m_n.

In general we cannot expect that $|\Pi^m_n| = \pi^m_n$ or $|\Sigma^m_n| = \sigma^m_n$ for $m,n > 0$. In order to use lemma 10.1 to show that $|\Pi^1_3| \leq \pi^1_3$, for example, we would want to show that for $\Gamma \in \Pi^1_3$, $X \in L_{\pi^1_3} \cap P(\omega) \Rightarrow \Gamma(X) \in L_{\pi^1_3}$. But there is no guarantee that $\Gamma^1 = \Gamma(\emptyset)$ belongs to $L_{\pi^1_3}$ or even to L. In the case of Π^1_1 and Σ^1_1, however, we can do better by making use of a result done to Barwise, Gandy and Moschovakis, formulated here in theorem 6.2.

Let In be the class of recursively inaccessible ordinals.

10.6. <u>Lemma</u>. If Γ is Π^1_1 or Σ^1_1 then $\langle\Gamma^\xi : \xi < \lambda\rangle$ is uniformly Σ^0_1 on L_λ for $\lambda \in \mathrm{In}$.

Proof. It is sufficient to show that if $\lambda \in \mathrm{In}$ then

(i) $x \in L_\lambda \Rightarrow \Gamma(x \cap \omega) \in L_\lambda$, and

(ii) $\{(x,y) \in L_\lambda x L_\lambda \mid \Gamma(x \cap \omega) = y\}$ is Σ_1^0 on L_λ uniformly for $\lambda \in \mathrm{In}$, as we may then carry through the proof of lemma 7.5.

Let Γ be Π_1^1. Then by 6.2(i) there is a Σ_1^{0-} formula φ^+ of $\mathcal{L}_\epsilon$ such that if A, B are admissible sets and $x \in A \in B$ then

$$n \in \Gamma(x \cap \omega) \Leftrightarrow B \models \varphi^+(A,n,x) .$$

But if $\lambda \in \mathrm{In}$ and $x \in L_\lambda$ then $x \in L_\mu$ for some admissible $\mu < \lambda$ and also $\mu^+ < \lambda$. So

$$n \in \Gamma(x \cap \omega) \Leftrightarrow L_{\mu^+} \models \varphi^+(L_\mu,n,x) .$$

Hence $\Gamma(x \cap \omega)$ is Σ_1^0 on L_{μ^+} so that $\Gamma(x \cap \omega) \in L_{\mu^++1} \subseteq L_\lambda$, proving (i).

To prove (ii) let σ_0 be the Π_3^{0-}sentence of 2.4. Then if $x,y \in L_\lambda$ and $\lambda \in \mathrm{In}$, $\Gamma(x \cap \omega) = y$ if and only if there are transitive set $A,B \in L_\lambda$ such that [A,B are transitive & $x \in A \in B$ & $A \models \sigma_0$ & $B \models \sigma_0$ & $\forall y \subseteq \omega$ & $\forall n \in \omega$ $(B \models \varphi^+(A,n,x) \Leftrightarrow n \in y)$] .

The expression [...] can be defined by a Σ_0^{0-}formula $\Psi(A,B,x,y)$ of $\mathcal{L}_\epsilon$, independent of λ, so that $\Gamma(x \cap \omega) = y \Leftrightarrow L_\lambda \models \exists a \exists b \Psi(a,b,x,y)$, proving (ii).

If Γ is Σ_1^1 then the proof is as above except that φ^+ is now Π_1^{0-}.

10.7. Theorem. $|\Pi_1^1| = \pi_1^1$ and $|\Sigma_1^1| = \sigma_1^1$.

Proof. $|\Pi_1^1| \geq \pi_1^1$ and $|\Sigma_1^1| \geq \sigma_1^1$ follows from 10.5. Note that by theorem 1.9. $\pi_1^1, \sigma_1^1 \in \mathrm{In}$, π_1^1 is Π_1^1-reflecting on In and σ_1^1 is Σ_1^1-reflecting on In. Hence by Lemma 10.6 we may use lemma 10.1 with $n = m = 1$, $X = \mathrm{In}$ and $A = \emptyset$ to get $|\Pi_1^1| \leq \pi_1^1$ and $|\Sigma_1^1| \leq \sigma_1^1$.

§ 11. Higher order inductive definitions, II.

Recall from the introduction that $\omega(\mathcal{R}) = \mathrm{Sup}\{|\prec| : \prec \in R$ well-orders a subset of $\omega\}$, where $|\prec|$ is the order type of the well-ordering $\prec$. In this section we will be concerned with comparing the ordinals $|\Pi^n_m|, |\Sigma^n_m|, |\Delta^n_m|, \pi^n_m, \sigma^n_m, \omega(\Pi^n_m), \omega(\Sigma^n_m)$ and $\omega(\Delta^n_m)$ when $n,m > 0$. We will prove theorems E, F and G of the introduction.

11.1. Theorem. Let $m,n > 0$.

(i) If $m+n > 2$ then $|\Delta^n_m| = \omega(\Delta^n_m)$.

(ii)
$$|\Delta^n_{m+1}| \begin{array}{l} > |\Pi^n_m| \geq \pi^n_m \geq \omega(\Sigma^n_m) \geq \\ \\ > |\Sigma^n_m| \geq \sigma^n_m \geq \omega(\Pi^n_m) \geq \end{array} \omega(\Delta^n_m)$$

(a) (b) (c) (d)

Proof.

(i) We first show that $|\Delta^n_m| \geq \omega(\Delta^n_m)$. Let $\prec$ be a Δ^n_m well-ordering of $A \subseteq \omega$. It suffices to find $\Gamma \in \Delta^n_m$ such that $|\Gamma| = |\prec|$. Let

$$\Gamma(X) = \{x \in A \,|\, \forall z(z \prec x \Longrightarrow z \in X)\} .$$

As $n > 0$ Γ is clearly Δ^n_m. By induction on α, $\Gamma^\alpha = \{x \in A : |x|_\prec < \alpha\}$ where $|x|_\prec$ is the order type of x in the well-ordering $\prec$. Hence $|\Gamma| = |\prec|$.

We next show that $\omega(\Delta^n_m) \geq |\Delta^n_m|$. The technique here is implicit in [14] . Let $\Gamma \in \Delta^n_m$ and

$$x \in Q \Longleftrightarrow x \in \Gamma^\infty \;\&\; \forall y < x(y \in \Gamma^\infty \Longrightarrow |y| \neq |x|) .$$

Then Q is a univalent system of notations for the ordinals less than $|\Gamma|$. Let

$$x \prec y \iff x,y \in Q \ \& \ |x| < |y| \ .$$

The $\prec$ is a well-ordering and $|\prec| = |\Gamma|$. It suffices to show that $\prec$ is Δ^n_m.

For $X \subseteq \omega \times \omega$ let $X_k = \{y : y X k\}$. If $S \subseteq \omega \times \omega$ and $Y \subseteq \omega$ let $\Phi(S,X,Y) \iff S$ is a (strict) well-ordering of Y & $\forall k \notin Y(X_k = \emptyset)$ & $\forall k \in Y(X_k = \bigcup\{\Gamma(X_l) : l S k\})$ & $\forall k,l \in Y(k \neq l \Rightarrow X_k \neq X_l)$ & $\Gamma(\bigcup_{k<\omega} X_k) \subseteq \bigcup_{k<\omega} X_k$.

Then as $m,n > 0$ Φ is Δ^n_m. Clearly $\Phi(S,X,Y)$ if and only if S is a well-ordering of Y of order type $|\Gamma|$ such that $X_k = \Gamma^{|k|_S}$ for $k \in Y$, and $X_k = \emptyset$ for $k \in Y$. Hence if

$$T(x,y) \iff x,y \in \Gamma^{\infty} \ \& \ |x| \leq |y| .$$

then

$$T(x,y) \iff \exists S \, \exists X \, \exists Y[\Phi(S,X,Y) \ \& \ \psi(X,x,y)]$$

$$\iff \forall S \forall X \forall Y[\Phi(S,X,Y) \Rightarrow \psi(X,x,y)]$$

where $\psi(X,x,y) \iff [x,y \in \bigcup_{k\in\omega} X_k \ \& \ \forall k(y \in X_k \Rightarrow x \in X_k)]$

Hence T is Δ^n_m. But

$$x \in Q \iff T(x,x) \ \& \ \forall y < x(T(y,y) \Rightarrow \neg(T(y,x) \ \& \ T(x,y))),$$

so that Q is Δ^n_m. As

$$x \prec y \iff x,y \in Q \ \& \ \neg T(y,x) \ ,$$

it follows that $\prec$ is Δ^n_m.

(ii) (a) Let Γ_1 be complete Σ^n_m such that $a \notin \Gamma_1^\infty$. Let $\Gamma_2(X) = \{a\}$ for all X and let $\Gamma = [\Gamma_1, \Gamma_2]$. Then $\Gamma \in \Delta^n_{m+1}$ and $|\Gamma| = |\Gamma_1| + 1 = |\Sigma^n_m| + 1$. Hence $|\Delta^n_{m+1}| > |\Sigma^n_m|$. $|\Delta^n_{m+1}| > |\Pi^n_m|$ is proved in the same way.

(b) is just 10.5. For (c) let $\prec$ be a Π^n_m well-ordering. Then there is a Σ^n_m sentence φ of $\mathcal{L}_P$ logically equivalent to

$$\exists f \forall k \forall l [k \prec l \Longrightarrow f(k) < f(l)] . \tag{1}$$

Then $\lambda \models \varphi \Longleftrightarrow \lambda \geq |\prec|$ and hence $|\prec|$ is not Σ^n_m reflecting. The proof that $\pi^n_m \geq \omega(\Sigma^n_m)$ is similar to the above, interchanging Π^n_m and Σ^n_m throughout and replacing (1) by

$$\neg \exists f \exists k \forall \alpha \forall \beta [\alpha < \beta \Longrightarrow f(\alpha) \prec f(\beta) \prec k] .$$

(d) is trivial.

<u>Remark</u>. We do not know of any cases where equality holds in (c). Note that $\pi^n_m \leq |\Pi^n_m| < \pi^n_{m+1}$. Thus π^n_m and $|\Pi^n_m|$ are not too far apart. Similarily for σ^n_m and $|\Sigma^n_m|$.

11.2. <u>Theorem</u>. $|\Delta^1_1|$ is not admissible.

<u>Proof</u>. We shall use the following fact extracted from § 7.10 of [19].

<u>Proposition</u>. There is a Π^1_1 relation $J \subseteq \omega \times \omega \times \mathcal{P}(\omega)$ such that Γ is Δ^1_1 if and only if $\Gamma = \Gamma_n = \{y : J(n,y,X)\}$ for some $n < \omega$.

11.3. <u>Lemma</u>: If λ is recursively inaccessible then $\langle \Gamma_n{}^\xi : n < \omega \,\&\, \xi < \lambda \rangle$ is Σ^0_1 on L_λ.

<u>Proof</u>. By 10.6, as each Γ_n is Π^1_1, $\langle \Pi_n{}^\xi : \xi < \lambda \rangle$ is Σ^0_1 on L_λ for all $n < \omega$. But as Γ_n is Π^1_1 uniformly in n, and the

proof of 10.6 is uniform, $\langle \Gamma_n{}^{\xi} : \xi < \lambda \rangle$ may be seen to be Σ_1^0 on L_λ uniformly in n giving the lemma.

11.4. Lemma. $|\Delta_1^1|$ is a limit of admissibles.

Proof. Let $\alpha < |\Delta_1^1|$. Then there is a $\Gamma \in \Delta_1^1$ such that $\alpha \leq |\Gamma|$. Let $\Theta(X) = \Xi\,(X) \cup \{\langle 3,a\rangle : a \in \Gamma(X)\}$, where Ξ is as in 8.16. Then $\mathcal{M} = \mathcal{M}^{\Theta}$ is a good notation system so that by the coding lemma $|M|$ is admissible. But as $\Gamma \leq_m \Theta$ it follows from 8.15 that $|\Gamma| \leq |M|$. But $|M| = |\Theta_{\leq}|$ and, as Γ is Δ_1^1, so is Θ and hence $\Theta_{\leq}$. Now let

$$\Lambda(X) = \Theta_{\leq}(X) \cup \{x : \Theta_{\leq}(X) \subseteq X \;\&\; x \in \omega\} .$$

Then Λ is Δ_1^1 and $|\Lambda| = |\Theta_{\leq}| + 1$. Hence $|M| < |\Lambda| \leq |\Delta_1^1|$, so that $\alpha \leq |M| < |\Delta_1^1|$ and $|M|$ is admissible proving the lemma. Note that we have shown that $|\Gamma| < |\Delta_1^1|$ for all $\Gamma \in \Delta_1^1$.

We can now prove the theorem. Suppose that $\lambda = |\Delta_1^1|$ is admissible. Then by the previous lemma it is recursively inaccessible and hence by lemma 11.3 $\langle \Gamma_n{}^{\xi} : n < \omega \;\&\; \xi < \lambda \rangle$ is Σ_1^0 on L_λ. Let $f(n) = |\Gamma_n|$ for $n < \omega$. Then $f : \omega \to \lambda$ is λ-recursive, as

$$f(n) = \mu\xi[\Gamma_n^{\xi+1} = \Gamma_n{}^{\xi}] .$$

But $\lambda = |\Delta_1^1| = \sup\limits_{n<\omega} |\Gamma_n| = \sup\limits_{n<\omega} f(n)$, contradicting the admissibility of λ.

We conclude this section by showing that under very general conditions $|\mathcal{C}| \neq |\neg\mathcal{C}|$. We also obtain a related spectrum result.

11.5. Definition. If $\mathcal{C}$ is a class of inductive definitions then the spectrum $\mathrm{sp}(\mathcal{C})$ of $\mathcal{C}$ is $\{|\Gamma| : \Gamma \in \mathcal{C}\}$.

11.6. Definition. A closed class $\mathcal{C}$ is first order closed if $\Gamma \in \mathcal{C}$ implies that $\Gamma_1, \Gamma_2 \in \mathcal{C}$ where

$$\Gamma_1(X) = \{e \mid \exists x\, \{e\}(x) \in \Gamma(X)\}$$

and

$$\Gamma_2(X) = \{e \mid \forall x\, \{e\}(x) \in \Gamma(X)\}\ .$$

11.7. Theorem. If $\mathcal{C}$ is first order closed then $|\mathcal{C}| \notin \mathrm{sp}(\neg\mathcal{C})$ and in particular $|\mathcal{C}| \neq |\neg\mathcal{C}|$.

Note that the last inequality follows because $|\neg\mathcal{C}| \in \mathrm{sp}(\neg\mathcal{C})$. As Π^n_m and Σ^n_m are easily seen to be first order closed for $m,n > 0$ we get

11.8. Corollary. If $m,n > 0$ then

$$|\Pi^n_m| \neq |\Sigma^n_m|.$$

We turn to the proof of the theorem. Let $\mathcal{C}$ be first order closed. Then $\neg\mathcal{C}$ is also first order closed. Let Γ be $\neg\mathcal{C}$-complete and let $\Delta \in \mathcal{C}$. It is sufficient to show that $|\Delta| \neq |\Gamma|$ as $|\Gamma| = |\neg\mathcal{C}|$. Let $\Theta(X) =$

$$\begin{aligned}
&\{\langle 1,a\rangle : a \in \Gamma(X)\}\\
&\cup\ \{\langle 2,a,b\rangle : a \in \mathcal{F}(X) \,\&\, \langle 4,b\rangle \in X_{<a}\}\\
&\cup\ \{\langle 3,a,b\rangle : \langle 6,a\rangle \in \mathcal{F}(X) \,\&\, b \notin \Delta(\{x: \langle 2,a,x\rangle \in \mathcal{F}(X)\})\}\\
&\cup\ \{\langle 4,a\rangle : \exists x[\langle 6,\langle 6,x\rangle\rangle \in \mathcal{F}(X) \,\&\, \langle 3,x,a\rangle \notin \mathcal{F}(X)]\}\\
&\cup\ \{a : a = \langle 5\rangle \,\&\, \forall x[x \in \Delta(\{y: \langle 4,y\rangle \in \mathcal{F}(X)\}) \Longrightarrow \langle 4,x\rangle \in \mathcal{F}(X)]\}\\
&\cup\ \{\langle 6,a\rangle : a \in \mathcal{F}(X)\}\ .
\end{aligned}$$

As $\neg\mathcal{C}$ is first order closed $\Theta \in \neg\mathcal{C}$. As $\Gamma \leq_m \Theta$ it follows that Θ is $\neg\mathcal{C}$-complete and hence by theorem 8.14

Lemma 1. $|\Gamma| = |M|$.

Lemma 2. For $\alpha \leq |M|$,

$$\langle 4,a\rangle \in M_\alpha \iff \exists\nu[\nu+3<\alpha \,\&\, a \in \Delta(\{y: \langle 4,y\rangle \in M_\nu\})] .$$

Proof.

$$\langle 2,x,y\rangle \in M_\nu \iff \exists\lambda < \nu. \quad \langle 2,x,y\rangle \in \Theta(M^*_\lambda)$$

$$\iff \exists\lambda < \nu. \quad \langle 6,x\rangle \in M_{\lambda+1} \,\&\, \langle 4,y\rangle \in M_{|x|}$$

$$\iff \langle 6,x\rangle \in M_\nu \,\&\, \langle 4,y\rangle \in M_{|x|} .$$

Now

$$\langle 6,x\rangle \in M_\lambda \iff \exists\sigma < \lambda. \quad \langle 6,x\rangle \in \Theta(M^*_\sigma)$$

$$\iff \exists\sigma < \lambda. \quad x \in M_\sigma$$

$$\iff |x| + 1 < \lambda .$$

Hence, letting

$\mathcal{Q}(a,\sigma) \iff a \in \Delta(\{y:\langle 4,y\rangle \in M_\sigma\})$, we have by ( ),

$$\langle 3,x,a\rangle \in M_\nu \iff \exists\lambda < \nu. \ \langle 6,x\rangle \in M_\lambda \,\&\, a \notin \Delta(\{y:\langle 2,x,y\rangle \in M_\lambda\})$$

$$\iff \exists\lambda < \nu. \ \langle 6,x\rangle \in M_\lambda \,\&\, a \notin \Delta(\{y:\langle 6,x\rangle \in M_\lambda \,\&\, \langle 4,y\rangle \in M_{|x|}\})$$

$$\iff \langle 6,\langle 6,x\rangle\rangle \in M_\nu \,\&\, \neg\, \mathcal{Q}(a,|x|) .$$

Hence

$$\langle 4,a\rangle \in M_\alpha \iff \exists\lambda < \alpha \ \exists x[\langle 6,\langle 6,x\rangle\rangle \in M_\lambda \,\&\, \langle 3,x,a\rangle \notin M_\lambda]$$

$$\iff \exists\lambda < \alpha \ \exists x[\langle 6,\langle 6,x\rangle\rangle \in M_\lambda \,\&\, [\langle 6,\langle 6,x\rangle\rangle \notin M_\lambda \vee \mathcal{Q}(a,|x|)]]$$

$$\iff \exists\lambda < \alpha \ \exists x[|x| + 2 < \lambda \,\&\, \mathcal{Q}(a,|x|)]$$

$$\iff \exists\nu[\nu + 3 < \alpha \,\&\, \mathcal{Q}(a,\nu)] \quad (\text{since } \alpha \leq |M|) .$$

Lemma 3. For $\alpha \leq |M|$ and $i < 4$, $\langle 4,a\rangle \in M_{4\alpha+i} \iff a \in \Delta^\alpha$.
In particular, $\Delta^\alpha \leq_m M_{4\alpha}$.

Proof: We use induction on α.

$$\langle 4,a\rangle \in M_{4\alpha+i} \iff \exists\nu[\nu+3 < 4\alpha+i \;\&\; \mathfrak{A}(a,\nu)]$$

$$\iff \exists\beta < \alpha . \exists j < 4\; \mathfrak{A}(a,4\beta+j)$$

$$\iff \exists\beta < \alpha . a \in \Delta\Delta^{\beta}$$

$$\iff a \in \Delta^{\alpha} .$$

Since $|\Gamma| = |M|$ it suffices to prove:

Lemma 4. $|\Delta| \neq |M|$.

Proof. Suppose $|\Delta| = |M| = \alpha$. We get a contradiction by showing $|\Delta| < \alpha$. Since $a \in M$ $\langle 6,a\rangle \in M$ & $|a|+1 = |\langle 6,a\rangle|$, α must be a limit ordinal and hence $\alpha = 4\alpha$. By definition of α , $\Delta\Delta^{\alpha} \subseteq \Delta^{\alpha}$. Hence by Lemma 3 ,

$$\forall x[x \in \Delta(\{a : \langle 4,a\rangle \in M_{\alpha}\}) \Rightarrow \langle 4,x\rangle \in M_{\alpha}]$$

i.e. $\langle 5\rangle \in M_{\alpha+1} \subseteq M$. Let $4\beta + i = |\langle 5\rangle| < \alpha$. Then by definition of M ,

$$\forall x[x \in \Delta(\{a : \langle 4,a\rangle \in M_{4\beta+i}\}) \Rightarrow \langle 4,x\rangle \in M_{4\beta+i}] .$$

By Lemma 3,

$$\forall x[x \in \Delta(\Delta^{\beta}) \Rightarrow x \in \Delta^{\beta}] ,$$

and hence $|\Delta| \leq \beta < \alpha$.

Appendix. Proof of the Coding Lemma.

§ A.1. Acceptable ordinal systems.

We begin by discussing certain closure properties on systems $\mathcal{M}$ and show that if $\mathcal{M}$ is a good notation system, then $\mathcal{M}$ has these closure properties.

A.1. Definition. Let $\mathcal{M} = \langle M, |\ | \rangle$ be any ordinal system and $B \subseteq \omega$. $\mathcal{M}$ is B-restricted productive if there is a primitive recursive function p such that for every $n \in \omega$,

(i) $\forall x[\{n\}(x,B) \in M] \implies p(n) \in M$ &

$$|p(n)| \geq \sup\{|\{n\}(x,B)|+1 : x \in \omega\} \ ;$$

(ii) $p(n) \in M \implies \forall x[\{n\}(x,B) \in M]$.

p is called a B-restriced productive function for $\mathcal{M}$.

The closure condition (i) is analogous to the closure of infinite regular cardinals with respect to mappings from smaller ordinals. (ii) is a technical requirement which ensures that there are not extraneous notations in M .

A.2. Definition. $\mathcal{M}$ is acceptable if there are recursive functions j, $\circledvee$ and a primitive recursive function p such that

(i) $j : M \leq_m M$ and for $a \in M$, if $|j(a)| \leq \alpha$ then $J(M_{|a|})$ is recursive in M_α uniformly in a ;

(ii) if $a \in M$ then $\lambda n.p(a,n)$ is an $M_{|a|}$-restricted productive function for $\mathcal{M}$.

(iii) $a \in M \vee b \in M \implies a \circledvee b \in M \ \& \ \inf\{|a|, |b|\} \leq |a \circledvee b|$.
We say that $\mathcal{M}$ is acceptable __in terms of__ j, p, $\circledvee$.

We next show that there are functions j, p, $\circledvee$ such that if $\mathcal{M}$ is a good notation system and λ is $\mathcal{M}$-good then $\mathcal{M}_\lambda$ is acceptable in terms of j, p, $\circledvee$.

A.3. __Lemma.__ Let $\mathcal{M} = \langle M, |\ | \rangle$ be a good notation system. If $a \in M$ then $J(M_{|a|})$ is recursive in M_α for all $\alpha \geq |a| + 2$, uniformly in a.

__Proof.__ Let $\alpha \geq |a| + 2$ and let e be a recursive function such that

$$[e(a,x)](t, M_{|a|}) = \begin{cases} a & \text{if } \ T^{M_{|a|}}(x,x,t) \\ 1 & \text{otherwise}. \end{cases}$$

Note that $1 \notin M$. It suffices to show there is a recursive function f such that for all x, $x \notin J(M_{|a|}) \iff f(a,x) \in M_\alpha$.
Now

$$\begin{aligned} x \notin J(M_{|a|}) &\iff \forall t \quad T^{M_{|a|}}(x,x,t) \\ &\iff \forall t[e(a,x)](t, M_{|a|}) = a \in M_{|a|+1} \\ &\iff \langle 1, a, e(a,x) \rangle \in M_\alpha . \end{aligned}$$

Thus let $f(a,x) = \langle 1, a, e(a,x) \rangle$.

To show that $\mathcal{M}$ satisfies A.2 (i) it remains to find a recursive function j so that for all $\mathcal{M}$-good λ, $j : M_\lambda \leq_m M_\lambda$ and for $a \in M_\lambda$, $|(a)| = |a| + 2$. Let e_1 be a recursive function such that for all a, t $[e_1(a)](t, M_{|0|}) = a$.

Then for $\mathcal{M}$-good λ,

$$a \in M_\lambda \implies \forall t\, [e_1(a)](t, M_{|0|}) = a \in M_\lambda$$
$$\implies \langle 1,0,e_1(a)\rangle \in M_\lambda \;\&\; |\langle 1,0,e_1(a)\rangle| = |a| + 1 .$$

Also $a \notin M_\lambda \implies \langle 1,0,e_1(a)\rangle \notin M_\lambda$. Thus let $j(a) = \langle 1,0,e_1(\langle 1,0,e_1(a)\rangle)\rangle$.

A. 4. <u>Lemma</u>. There are functions j, p, $\circledvee$ such that if $\mathcal{M}$ is a good notation system and λ is $\mathcal{M}$-good, then $\mathcal{M}_\lambda$ is acceptable in terms of j, p, $\circledvee$.

<u>Proof</u>. It remains to find $\circledvee$ and p. It is easy to see that $a \circledvee b = \langle 2,a,b\rangle$ has the desired property. To find p, let e be a primitive recursive function such that for any n and $X \subseteq \omega$, the range of $\lambda t[e(n)](t,X)$ equals $\{0\} \cup$ the range of $\lambda t.\{n\}(t,X)$. Then for $a \in M_\lambda$,

$$\forall x\{n\}(x, M_{|a|}) \in M_\lambda \iff \forall x[e(n)](x, M_{|a|}) \in M_\lambda$$

and if $\forall x\{n\}(x,M_{|a|}) \in M_\lambda$ then

$$\sup\{|\{n\}(x,M_{|a|})| : x \in \omega\} = \sup\{|[e(n)](x,M_{|a|})| : x \in \omega\} .$$

Thus it is easy to see that we can choose $p(a,n) = \langle 1,a,e(n)\rangle$.

In view of Lemma A.4, to prove the Coding Lemma it suffices to prove the following:

A. 5. <u>Theorem</u>. Let $\mathcal{M} = \langle M, |\,|\rangle$ be acceptable in terms of $g, p, \circledvee$.

(i) $|M| \in \mathrm{Ad}(T_{\mathcal{M}})$;

(ii) For every Σ_1^{0-} formula $\varphi(v_1,\ldots,v_n)$ of $\mathcal{L}_p(T_{\mathcal{M}})$, there is a primitive recursive function h such that

$a_1,\dots,a_n \in M$ & $|M| \models \varphi(|a_1|,\dots,|a_n|)$ iff $h(a_1,\dots,a_n) \in M$;

Furthermore h is completely determined by the functions j, p, $\textcircled{v}$, a member $u_o \in M$, and the formula φ .

(iii) Let F be an ordinal function which is $|M|$-partial recurcive in $T_{\mathfrak{M}}$. Then there is a recursive function k such that for $a_1,\dots,a_n \in M$, if $F(|a_1|,\dots,|a_n|)$ is defined then $k(a_1,\dots,a_n) \in M$ and $F(|a_1|,\dots,|a_n|) \leq |k(a_1,\dots,a_n)|$.

(iv) $X \subseteq \omega$ is $|M|$-r.e. in $T_{\mathfrak{M}}$ iff $X \leq_m M$.

The remainder of the appendix is devoted to the proof of Theorem A.5. Suppose $\mathfrak{M} = \langle M, |\,| \rangle$ is acceptable in terms of j, p, $\textcircled{v}$. Let $u_o \in M$ and $|u_o| = 0$. In Lemmas A. 6-16 below the reader should observe that the functions described are either independent of the particular acceptable system $\mathfrak{M}$ or are completely determined by j, p, $\textcircled{v}$ and u_o . (In Lemma A.8 the choice of e is independent of $\mathfrak{M}$; in Lemma A.9 an index of h can be found as the value of a recursive function of the indices of the f_i, g_i which is independent of $\mathfrak{M}$.)

A. 6. <u>Lemma</u>. There is a recursive function $+_M$ such that:

(i) $a \in M$ & $b \in M \implies a +_M b \in M$ & $|a +_M b| > \max\{|a|,|b|\}$;

(ii) $a +_M b \in M \implies a \in M$ & $b \in M$.

<u>Proof</u>. Let e be a recursive function such that

$$\{e(a,b)\}(t,M_o) = \begin{cases} a & \text{if } t = 0 \\ b & \text{otherwise,} \end{cases}$$

and let $a +_M b = p(u_o, e(a,b))$.

§ A.2. $\mathcal{M}$ -recursion.

We next define a class of partial number-theoretic functions based on $\mathcal{M}$. These functions behave very much like the functions partial recursive in the type 2 functional E of Kleene [9], where for $f \in {}^{\omega}\omega$,

$$E(f) = \begin{cases} 0 & \text{if } \exists t[f(t)=0] , \\ 1 & \text{otherwise} . \end{cases}$$

Using these functions we are able to carry out computations involving $\mathcal{M}$ which are needed to show that $|M|$ is admissible. The following definition by schemata of the predicate $\{z\}^{\mathcal{M}}(\vec{x}) \simeq y$ parallels the corresponding definition by Kleene [9] of the partial recursive functionals of finite types. As described in [9] this definition by schemata may be viewed as a transfinite inductive definition. The essential difference here is that there are an infinite number of starting functions in the case S0 . Thus the characteristic function of each M_α for $\alpha < |M|$ is given outright.

In the following, $\vec{x}$ and $\vec{y}$ are abbreviations for $x_1,\ldots,x_n$ and $y_1,\ldots,y_m$, respectively.

S0.a $\quad \{\langle 0,1,a\rangle\}^{\mathcal{M}}(x) = \begin{cases} 0 & \text{if } x \in M_{|a|} , \\ 1 & \text{otherwise} . \end{cases}$ for each $a \in M$;

S1. $\quad \{\langle 1,n\rangle\}^{\mathcal{M}}(\vec{x}) = x_1 + 1$;

S2. $\quad \{\langle 2,n,q\rangle\}^{\mathcal{M}}(\vec{x}) = q$;

S3. $\quad \{\langle 3,n\rangle\}^{\mathcal{M}}(\vec{x}) = x_1$;

S4. $\quad \{\langle 4,n,a,b\rangle\}^{\mathcal{M}}(\vec{x}) \simeq \{a\}^{\mathcal{M}}(\{b\}^{\mathcal{M}}(\vec{x}),\vec{x})$.

S5. $\begin{cases} \{\langle 5,n+1,a,b\rangle\}^{\mathcal{M}}(0,\vec{x}) \simeq \{a\}^{\mathcal{M}}(\vec{x}) \\ \{\langle 5,n+1,a,b\rangle\}^{\mathcal{M}}(y+1,\vec{x}) \simeq \{b\}^{\mathcal{M}}(y,\{\langle 5,n+1,a,b\rangle\}^{\mathcal{M}}(y,\vec{x}),\vec{x}) \end{cases}$

S6. $\{\langle 6,n,k,a\rangle\}^{\mathcal{M}}(\vec{x}) \simeq \{a\}^{\mathcal{M}}(\vec{x}_1)$,

where $\vec{x}_1$ is obtained from $\vec{x}$ by moving x_{k+1} to the front.

S8. $\{\langle 8,n,a\rangle\}^{\mathcal{M}}(\vec{x}) \simeq E(\lambda t.\{a\}^{\mathcal{M}}(t,\vec{x}))$,

where both sides are undefined if for the given $\vec{x}$, $\lambda t.\{a\}^{\mathcal{M}}(t,\vec{x})$ is not total.

S9. $\{\langle 9,n+m+1,m\rangle\}^{\mathcal{M}}(z,\vec{x},\vec{y}) \simeq \{z\}^{\mathcal{M}}(\vec{x})$.

Since we are defining only partial <u>number-theoretic</u> functions there is no S7 clause. $\{z\}^{\mathcal{M}}$ is called the $\mathcal{M}$-<u>partial recursive function with index</u> z . $\{z\}^{\mathcal{M}}$ is $\mathcal{M}$-<u>recursive</u> if it is everywhere defined. Note that if z is the index of an $\mathcal{M}$-partial recursive function, then $(z)_1$ is the number of variables of the function. It is easy to prove the standard theorems of recursive function theory with the exception of the normal form theorem. In particular the Kleene S-m-n theorem, the Kleene second recursion theorem and the theorem on definition by cases are proved exactly as in [9].

Thus we have:

A. 7. <u>Lemma</u>. For each $m \geq 1$: There is a primitive recursion function $S^m(z,y_1,\ldots,y_m)$ such that if $f(y_1,\ldots,y_m,\vec{x})$ is an $\mathcal{M}$-partial recursive function with index z then for each fixed $y_1,\ldots,y_m$, $S^m(z,y_1,\ldots,y_m)$ is an index of $f(y_1,\ldots,y_m,\vec{x})$ as a function of $\vec{x}$.

A. 8. Lemma. (Second recursion theorem). Given any $\mathcal{M}$-partial recursive function $f(z,\vec{x})$ an integer e can be found such that $\{e\}^{\mathcal{M}}(\vec{x}) \simeq f(e,\vec{x})$.

A.9. Lemma. If f_0, f_1, g_0, g_1 are $\mathcal{M}$-partial recursive and $\{\vec{x} : g_0(\vec{x}) = 0\} \cap \{\vec{x} : g_1(\vec{x}) = 0\} = \emptyset$
then the function

$$h(\vec{x}) \simeq \begin{cases} f_0(\vec{x}) & \text{if } g_0(\vec{x}) \simeq 0 , \\ f_1(\vec{x}) & \text{if } g_1(\vec{x}) \simeq 0 . \end{cases}$$

is $\mathcal{M}$-partial recursive

A.10. Lemma. If f is $\mathcal{M}$-partial recursive, then so is $\mu y[f(\vec{x},y) \simeq 0]$.

A.11. Remark. It is clear from scheme S0 and the fact that the $\mathcal{M}$-partial recursive functions are closed under composition, primitive recursion and the μ-scheme, that each function partial recursive in some M_α , where $\alpha < |M|$, is $\mathcal{M}$-partial recursive.

It is convenient to deal with functions of just one variable. Let $\{z\}^{\mathcal{M}}[a] \simeq \{z\}^{\mathcal{M}}((a_0),\ldots,(a)_{(z)_1-1})$ and let $D = \{\langle z,a\rangle : \{z\}^{\mathcal{M}}[a]$ is defined$\}$. The inductive definition described by schemata S0-S9 assoziates with each $\langle z,a\rangle \in D$ an ordinal as follows. $|\langle z,a\rangle|^{\mathcal{M}} = 0$ if $(z)_0$ is $0,1,2,$ or 3 , that is if $\{z\}^{\mathcal{M}}((a)_0,\ldots,(a)_{(z)_1-1})$ is defined by one of S0-S3. In case S4, letting $\vec{a} = (a)_0,\ldots,(a)_{(z)_1-1}$,

$$\{z\}^{\mathcal{M}}[a] = \{z\}^{\mathcal{M}}(\vec{a}) = \{b\}^{\mathcal{M}}(\{c\}^{\mathcal{M}}(\vec{a}),\vec{a}) = \{b\}^{\mathcal{M}}[\langle\{c\}^{\mathcal{M}}[a],\vec{a}\rangle]$$

for some b,c . Thus let

$$|\langle z,a\rangle|^{\mathcal{M}} = \max\{|\langle c,a\rangle|^{\mathcal{M}}, |\langle b,\langle\{c\}^{\mathcal{M}}[a],\vec{a}\rangle\rangle|^{\mathcal{M}}\}+1 .$$

The cases S5, S6, and S9 are similar. For example in case S9,

$$\{u\}^{\mathcal{M}}[\langle z,\vec{a},\vec{y}\rangle] = \{u\}^{\mathcal{M}}(z,\vec{a},\vec{y}) = \{z\}^{\mathcal{M}}(\vec{a}) = \{z\}^{\mathcal{M}}[a] .$$

Thus let $|\langle u,\langle z,\vec{a},\vec{y}\rangle\rangle|^{\mathcal{M}} = |\langle z,a\rangle|^{\mathcal{M}} + 1$. In case S8 we have

$$\{z\}^{\mathcal{M}}[a] = \{z\}^{\mathcal{M}}(\vec{a}) = E(\lambda t.\{b\}^{\mathcal{M}}(t,\vec{a})) = E(\lambda t.\{b\}^{\mathcal{M}}[\langle t,\vec{a}\rangle])$$

for some b . In this case we define

$$|\langle z,a\rangle|^{\mathcal{M}} = \sup\{|\langle b,\langle t,\vec{a}\rangle\rangle|^{\mathcal{M}} + 1 : t \in \omega\} .$$

The ordinal function $|\ |^{\mathcal{M}}$ makes possible proofs by induction. The following lemma and corollary are a generalization of the fact that a function recursive in E is actually recursive in 0_α for some $\alpha < \omega_1$ (where 0 is from Kleene [22]).

A.12. <u>Lemma</u>. There are recursive functions f and g such that

(i) $\langle z,a\rangle \in D \iff g(z,a) \in M$;

(ii) If $\langle z,a\rangle \in D$ then for all x ,
$$\{z\}^{\mathcal{M}}[a] = \{f(z,a)\}(x,M_{|g(z,a)|}) .$$

<u>Proof</u>. The recursive functions f and g are defined simultaneously by the Kleene second recursion theorem of ordinary recursive function theory. (ii) and (i) in the direction $\Longrightarrow$ are then proven by induction on $|\langle z,a\rangle|^{\mathcal{M}}$. (i) in the direction $\Longleftarrow$ is proven by induction on $|g(z,a)|$. A rigorous proof would require an elaborate definition of f and g involving a number of auxiliary functions arising from applications of the S-m-n theorem of ordinary recursive function theory. Instead of this we give an informal description suppressing explicit reference to most of the auxiliary functions. We begin by assuming $\langle z,a\rangle \in D$ and

show in case Si how $f(z,a)$ and $g(z,a)$ must be defined in this case so that they satisfy (ii) and (i) in the direction $\Longrightarrow$. Then we show in S'i that if $f(z,a)$ and $g(z,a)$ are defined as in Si then $g(z,a) \in M$ implies $\langle z,a\rangle \in D$. The reader familiar with the second recursion theorem will have no trouble in showing, if desired, that there actually exist such recursive functions f and g.

Case S0. $\langle z,a\rangle \in D$ and

$$\{z\}^{\mathfrak{M}}[a] = \{\langle 0,1,(z)_2\rangle\}^{\mathfrak{M}}((a)_0)$$

$$= \begin{cases} 0 & \text{if } (a)_0 \in M_{|(z)_2|}, \\ 1 & \text{otherwise}. \end{cases}$$

Thus let $g(z,a) = (z)_2$ and choose $f(z,a)$ so that for all x,

$$\{f(z,a)\}(x,M_{|(z)_2|}) = \begin{cases} 0 & \text{if } (a)_0 \in M_{|(z)_2|}, \\ 1 & \text{otherwise}. \end{cases}$$

Case S'0. $z = \langle 0,1,(z)_2\rangle$ and $f(z,a) = (z)_2 \in M$. Since $(z)_2 \in M$, $\{z\}^{\mathfrak{M}}[a] \simeq \{\langle 0,1(z)_2\rangle\}^{\mathfrak{M}}((a)_0)$ is defined by clause S0 in the definition of $\{\ \}^{\mathfrak{M}}$.

The definition of f and g is trivial in cases S1-S3, and easy in cases S5, S9. We consider in detail cases S8 and S4.

Case S8. $\langle z,a\rangle \in D$ and

$$\{z\}^{\mathfrak{M}}[a] = E(\lambda t.\{b\}^{\mathfrak{M}}[\langle t,\vec{a}\rangle]) = \begin{cases} 0 & \text{if } \quad t.\{b\}^{\mathfrak{M}}[\langle t,\vec{a}\rangle] = 0, \\ 1 & \text{otherwise}, \end{cases}$$

where $b = (z)_2$. By the induction hypothesis,

(1) $\forall t[g(b,\langle t,\vec{a}\rangle) \in M]$ and for all t, x,

(2) $\{b\}[\langle t,\vec{a}\rangle] = \{f(b,\langle t,\vec{a}\rangle)\}(x,M_{|g(b,\langle t,\vec{a}\rangle)|})$.

Since $\mathcal{M}$ is $\emptyset$-restricted productive we can find $u \in M$ such that for all t, $|jg(b,\langle t,\vec{a}\rangle)| < |u|$. (More precisely $u = p(0,e)$ where for all t, $\{e\}(t,M_{|0|}) = jg(b,\langle t,\vec{a}\rangle)$. e of course depends on g, a and b.) Then by A.2. (i),

$$(3) \qquad M_{|g(b,\langle t,\vec{a}\rangle)|} \leq_t J(M_{|g(b,\langle t,\vec{a}\rangle)|}) \leq_t M_{|u|} \leq_t J(M_{|u|}) \leq_t M_{|j(u)|},$$

where these reducibilities are uniform in a, b, t. Then from (2), (3) we can find a v (depending on a, b) so that for all t, $\{b\}^{\mathcal{M}}[\langle t,\vec{a}\rangle] = \{v\}(t,M_{|u|})$. Then choose w so that

$$(4) \qquad E(\lambda t.\{b\}^{\mathcal{M}}[\langle t,\vec{a}\rangle]) = 0 \Leftrightarrow \exists t.\{b\}^{\mathcal{M}}[\langle t,a\rangle] = 0 \Leftrightarrow \exists t.\{v\}(t,M_{|u|}) = 0 \Leftrightarrow w \in J(M_{|u|}).$$

Let $g(z,a) = j(u)$. Then choose $f(z,a)$ so that for all x,

$$\{f(z,a)\}(x,M_{|j(u)|}) = \begin{cases} 0 & \text{if } w \in J(M_{|u|}), \\ 1 & \text{otherwise}. \end{cases}$$

It follows from (4) that $f(z,a)$ satisfies the desired equation.

Case S'8. $z = \langle 8,n,b\rangle$ and $g(z,a) \in M$. Since $g(z,a) = j(u)$, $j(u) \in M$; since $j{:}M \leq_m M$, $u \in M$; since $u = p(0,e) \in M$ we have by A.2 (ii), for all t, $\{e\}(t,M_{|0|}) = jg(b,\langle t,\vec{a}\rangle) \in M$ and hence $g(b,\langle t,\vec{a}\rangle) \in M$. Also by A.2 (i), (ii), for all t,

$$|g(b,\langle t,\vec{a}\rangle)| < |jg(b,\langle t,\vec{a}\rangle)| < |u| < |j(u)| = |g(z,a)|.$$

Hence by the induction hypothesis, for all t, $\langle b,\langle t,\vec{a}\rangle\rangle \in D$, i.e. $\{b\}^{\mathcal{M}}[\langle t,\vec{a}\rangle]$ is defined. Hence $\{z\}^{\mathcal{M}}[a] \simeq E(\lambda t.\{b\}^{\mathcal{M}}[\langle t,\vec{a}\rangle])$ is defined, i.e. $\langle z,a\rangle \in D$.

Case S4. $\langle x,a\rangle \in D$ and $\{z\}^{\mathcal{M}}[a] = \{b\}^{\mathcal{M}}[\langle \{c\}^{\mathcal{M}}[a],\vec{a}\rangle]$.
Let $D = \langle \{c\}[a],\vec{a}\rangle$. Then $\langle c,a\rangle \in D$ and $\langle b,d\rangle \in D$, and $|\langle c,a\rangle|^{\mathcal{M}}, |\langle b,d\rangle|^{\mathcal{M}} < |\langle z,a\rangle|^{\mathcal{M}}$. By the induction hypothesis,

$$g(c,a) \in M \quad \text{and} \quad g(b,d) \in M ,$$

and for all x ,

(5) $$\{c\}^{\mathcal{M}}[a] = \{f(c,a)\}(x,M_{|g(c,a)|}) , \quad \text{and}$$
$$\{z\}^{\mathcal{M}}[a] = \{b\}^{\mathcal{M}}[d] = \{f(b,d)\}(x,M_{|g(b,d)|}) .$$

We begin by showing how to choose $g(z,a)$ so that $g(z,a) \in M$ and

(6) $$|jg(c,a)|, |jg(b,d)| < |g(z,a)| .$$

By we can find a u (depending on a,b,c) such that for every x,

(7) $$jg(b,d) = jg(b,\langle \{f(c,a)\}(x,M_{|g(c,a)|}),\vec{a}\rangle)$$
$$= \{u\}(x,M_{|g(c,a)|}) .$$

By A.2 (ii), $p(g(c,a),u) \in M$ and

(8) $$|p(g(c,a),u)| \geq \sup\{|\{u\}(x,M_{|g(c,a)|})| + 1 : x \in \omega\}$$
$$= |jg(b,d)| + 1 .$$

Let $g(z,a) = p(g(c,a),u) +_M jg(c,a)$. It is clear from (8) that $g(z,a)$ satisfies (6). To find $f(z,a)$ observe that by (6) and A.2 (i),

(9) $$M_{|g(b,d)|} \leq_t J(M_{|g(b,d)|}) \leq_t M_{|g(z,a)|} ,$$

uniformly in b,d ; and

(10) $$M_{|g(c,a)|} \leq_t J(M_{|g(c,a)|}) \leq_t M_{|g(z,a)|} ,$$

uniformly in c,a . From (5), (9) we can find v, w, y so that for all x ,

$$\begin{aligned}\{z\}^{\mathcal{M}}[a] &= \{f(b,d)\}(x,M_{|g(b,d)|}) \\ &= \{v\}(x,b,d,M_{|g(z,a)|}) \\ &= \{w\}(x,b,\{f(c,a)\}(x,M_{|g(c,a)|}),M_{|g(z,a)|}) \\ &= \{y\}(x,M_{|g(z,a)|}) ,\end{aligned}$$

where w is obtained by eliminating d in the previous equation by referring back to the definition of d and then using (10), and y is obtained by using (10). Thus let $f(z,a) = y$.

<u>Case</u> S'4. $z = \langle 4,n,a,b\rangle$ and $g(z,a) \in M$. Since

$$g(z,a) = p(g(c,a),u) +_M jg(c,a) \in M$$

we have $g(c,a),p(g(c,a),u) \in M$ and $|g(c,a)|,|p(g(c,a),u)| < |g(z,a)|$. By the induction hypothesis, $\{c\}^{\mathcal{M}}[a]$ is defined. Also by (7),

$$jg(b,d) = \{u\}(0,M_{|g(c,a)|}) \in M ,$$

and hence $g(b,d) \in M$ and $|g(b,d)| < |p(g(c,a),u)| < |g(z,a)|$. Again by the induction hypothesis, $\{b\}^{\mathcal{M}}[d]$ is defined. Since $\{z\}^{\mathcal{M}}[a] \simeq \{b\}^{\mathcal{M}}[d]$, $\{z\}^{\mathcal{M}}[a]$ is defined.

$X \subseteq {}^n\omega$ is said to be $\mathcal{M}$-<u>r.e.</u> if X is the domain of an $\mathcal{M}$-partial recursive function; X is $\mathcal{M}$-<u>recursive</u> if the representing function of X is $\mathcal{M}$-recursive.

A.13. <u>Corollary</u>. Let $X \subseteq \omega$.

(i) X is $\mathcal{M}$-r.e. iff $X \leq_m M$;

(ii) X is $\mathcal{M}$-recursive iff $X \leq_t M_\alpha$ for some $\alpha < |M|$;

(iii) If h is $\mathcal{M}$-partial recursive there is a recursive function k such that for all $\vec{a}$,

$$h(\vec{a}) \in M \implies k(\vec{a}) \in M \ \& \ |h(\vec{a})| \leq |k(\vec{a})| \ ;$$

(iv) If $h : \omega \to M$ and h is $\mathcal{M}$-recursive, then $\sup\{|h(x)| : x \in \omega\} < |M|$.

<u>Proof</u>. (i) If X is $\mathcal{M}$-r.e. then there is a $z \in \omega$ such that $n \in X \iff \{z\}^{\mathcal{M}}(n)$ is defined. So if g is as in lemma A.12. then $n \in X \iff g(z,\langle n\rangle) \in M$.

Thus $X \leq_m M$. Now suppose h is a recursive function such that $h : X \leq_m M$ and choose e so that $\{e\}^{\mathcal{M}}(n) \simeq \{\langle 0,1,h(n)\rangle\}^{\mathcal{M}}(0)$. Then,

$$\begin{aligned} n \in X &\iff h(n) \in M \\ &\iff \{\langle 0,1,h(n)\rangle\}^{\mathcal{M}}(0) \text{ is defined} \\ &\iff \{e\}^{\mathcal{M}}(n) \text{ is defined.} \end{aligned}$$

Thus X is $\mathcal{M}$-r.e.

(ii) It follows from Remark A.11 that if X is recursive in M_α for some $\alpha < |M|$, then X is $\mathcal{M}$-recursive. For the other direction it suffices to show that each total function $\{z\}^{\mathcal{M}}$ is recursive in M_α for some $\alpha < |M|$. By A.12., for all n ,

$$(11) \qquad \{z\}^{\mathcal{M}}(n) = \{z\}^{\mathcal{M}}[\langle n\rangle] = \{f(z,\langle n\rangle)\}(0,M_{|g(z,\langle n\rangle)|}) \ .$$

Choose e so that for all n , $\{e\}(n,M_{|u_0|}) = jg(z,n)$ and let $c = p(u_0,e)$. Then $c \in M$ and $|c| > |jg(z,n)|$ for all n . Hence for all n , $M_{|g(z,n)|} \leq_t M_{|c|}$ uniformly in n . Thus there is an e_1 such that for all n ,

$$\{e_1\}(n,M_{|c|}) = \{f(z,\langle n\rangle)\}(0,M_{|g(z,\langle n\rangle)|}) \ .$$

Then from (11), $\{z\}^{\mathcal{M}}$ is recursive in $M_{|c|}$.

(iii) Let $h = \{z\}^{\mathcal{M}}$, $a = \langle\vec{a}\rangle$, and $k(\vec{a}) = p(g(z,a), f(z,a))$.

For $h(\vec{a}) \in M$, $\{z\}^{\mathcal{M}}[a] = h(\vec{a}) \in M$; hence by A.12, $g(z,a) \in M$ and for all x,

$$\{z\}^{\mathcal{M}}[a] = \{f(z,a)\}(x, M_{|g(z,a)|}) \in M .$$

Thus $k(\vec{a}) \in M$ and

$$|k(\vec{a})| \geq \sup\{|\{f(z,a)\}(x, M_{|g(z,a)|})| + 1 : x \in \omega\} = |h(\vec{a})| + 1 .$$

(iv) Let k be as in (iii). Then for all x, $k(x) \in M$ and $|h(x)| < |k(x)|$. Choose e so that for all x, $k(x) = \{e\}(x, M_{|u_0|})$. Then $p(u_0, e) \in M$ and for all x, $|h(x)| < |k(x)| < |p(u_0,e)|$.

§ A.3. Selection.

Using (V) and the fact that by SO every M_α is $\mathcal{M}$-recursive uniformly in a notation for α, guven $a \in M \vee b \in M$ it is possible to decide $\mathcal{M}$-recursively whether $|a| \leq |b|$ or $|b| < |a|$. (Recall that if $a \notin M$ then $|a| = |M|$.) More precisely:

A.14. Lemma. There is an $\mathcal{M}$-partial recursive function d such that:

(i) $a \in M \ \& \ |a| \leq |b| \implies d(a,b) = 0$;

(ii) $|b| < |a| \implies d(a,b) = 1$.

Proof. Let $k(x) = \langle 0,1,x \rangle$ and

$$d(a,b) \simeq \begin{cases} 0 & \text{if } \{kh(a,b)\}^{\mathcal{M}}(a) \simeq 0 \ \& \ \{k(a)\}^{\mathcal{M}}(b) \simeq 1 ; \\ 1 & \text{if } [\{kh(a,b)\}^{\mathcal{M}}(a) \simeq 0 \ \& \ \{k(a)\}^{\mathcal{M}}(b) \simeq 0] \\ & \quad \vee \{kh(a,b)\}^{\mathcal{M}}(a) \simeq 1 . \end{cases}$$

Using A.9 it is easy to see that d is $\mathcal{M}$-partial recursive.

Note that if $\neg(a \in M \vee b \in M)$ then $d(a,b)$ is not defined. If $a \in M$ & $|a| \leq |b|$ then $h(a,b) \in M$ and $|a| \leq |a \circledV b| < |h(a,b)|$; Hence $a \in M_{|h(a,b)|}$ & $b \notin M_{|a|}$ and so $\{kh(a,b)\}^{\mathcal{M}}(a) = 0$ & $\{k(a)\}^{\mathcal{M}}(b) = 1$. Thus $d(a,b) = 0$. Similarly, if $|b| < |a|$ then $h(a,b) \in M$ and either $a \in M_{|h(a,b)|}$ & $b \in M_{|a|}$ or $a \notin M_{|h(a,b)|}$ and hence by the definition of d , $d(a,b) = 1$.

The following argument is similar to Gandy's unpublished proof of the existence of selection functions associated with functionals of type 2.

A.15. Lemma. There is an $\mathcal{M}$-partial recursive function v such that if $\lambda t.\{z\}(t)$ is total then

$$\exists t\{z\}(t) \in M \implies v(z) \text{ is defined } \& \ \{z\}(v(z)) \in M .$$

Proof. Let g be the recursive function of A.12 and let e be obtained from the second recursion theorem (Lemma A.8) so that

$$(12) \quad \{e\}^{\mathcal{M}}(t,z) \simeq \begin{cases} 0 & \text{if } \{z\}(t) \in M \\ \{e\}^{\mathcal{M}}(t+1,z)+1 & \text{if } d(\{z\}(t), g(e,\langle t+1,z\rangle)) = 1 . \end{cases}$$

e is found by using the recursion theorem in a manner similar to the proof of Kleene [9 ,XVI]. Let y be the least t such that $\{z\}(t) \in M$; equivalently, y is the least t such that $\{e\}^{\mathcal{M}}(t,z) = 0$. We show by induction on x that $\{e\}^{\mathcal{M}}(y-x,z) = x$ for $0 \leq x \leq y$. This is true if $x = 0$ since in this case $\{e\}^{\mathcal{M}}(y-x,z) = \{e\}^{\mathcal{M}}(y,z) = 0 = x$. Suppose $x > 0$. Then by the induction hypothesis $\{e\}^{\mathcal{M}}(y-(x-1),z) = x-1$. In particular, since $\{e\}^{\mathcal{M}}(y-(x-1),z)$ is defined, $g(e,\langle y-(x-1),z\rangle) \in M$. Since also $\{z\}(y-x) \notin M$, we have $d(\{z\}(y-x), g(e,\langle y-(x-1),z\rangle)) = 1$. This implies by (12),

$$\{e\}^{\mathfrak{M}}(y-x,z) = \{e\}^{\mathfrak{M}}(y-(x-1),z)+1 = (x-1)+1 = x \ .$$

Setting $x = y$ this gives $\{e\}^{\mathfrak{M}}(0,z) = y$. Thus it suffices to let $v(z) \simeq \{e\}^{\mathfrak{M}}(0,z)$.

A.16. Corollary. Let $Q \subseteq {}^{n+1}\omega$ be $\mathfrak{M}$-r.e. Then there is an $\mathfrak{M}$-partial recursive function $\lambda\vec{x}\nu yQ(\vec{x},y)$ of n variables such that for all $\vec{x}$,

$$\exists yQ(\vec{x},y) \Longrightarrow \nu yQ(\vec{x},y) \text{ is defined } \& \ Q(\vec{x},\nu yQ(\vec{x}.y)) \ .$$

Proof. Let $Q(\vec{x},y) \Longleftrightarrow \{z\}^{\mathfrak{M}}(\vec{x},y)$ is defined. Then

$$Q(\vec{x},y) \Longleftrightarrow \{S^m(z,\vec{x})\}^{\mathfrak{M}}(y) \text{ is defined}$$

$$\Longleftrightarrow g(S^m(z,\vec{x}),\langle y\rangle) \in M \ .$$

Let e be a recursive function such that $\{e(\vec{x})\}(y) = g(S^m(z,\vec{x}),\langle y\rangle)$. Then

$$\exists yQ(\vec{x},y) \Longleftrightarrow \exists y\{e(\vec{x})\}(y) \in M \ .$$

Thus let $\nu yQ(\vec{x},y) \simeq de(\vec{x})$.

The following lemma summarises some of the properties of $\mathfrak{M}$-r.e. relations.

17. Lemma. (i) If f is $\mathfrak{M}$-partial recursive, then the relation $f(\vec{x}) = z$ is $\mathfrak{M}$-r.e.

(ii) If Q is $\mathfrak{M}$-r.e., then the function

$$f(\vec{x}) \simeq \begin{cases} z & \text{if } Q(\vec{x}) \ , \\ \text{undefined otherwise} \ , \end{cases}$$

is $\mathfrak{M}$-partial recursive.

(iii) The relations $y \in M \ \& \ |x| < |y|$; $y \in M \ \& \ |x| \leq |y|$,

$y \in M$ & $|y| \leq |x|$ are $\mathcal{M}$-r.e.

(iv) The $\mathcal{M}$-r.e. relations are closed under conjunction, disjunction, universial and existential number quantification, and inverse images by $\mathcal{M}$-recursive functions.

Proof. Suppose f is $\mathcal{M}$-partial recursive. Let

$$e(x,z) = \begin{cases} \langle 2,1,0 \rangle & \text{if } x = z , \\ 0 & \text{otherwise} , \end{cases}$$

and $g(\vec{x},z) \simeq \{e(f(\vec{x}),z)\}^{\mathcal{M}}(0)$. Then g is $\mathcal{M}$-partial recursive and

$$\begin{aligned} g(\vec{x},z) \text{ is defined} &\Longleftrightarrow e(f(\vec{x}),z)) = \langle 2,1,0 \rangle \\ &\Longleftrightarrow f(\vec{x}) = z . \end{aligned}$$

To prove (ii) let $Q(\vec{x}) \Longleftrightarrow g(\vec{x})$ is defined, and let $f(\vec{x}) \simeq 0 \cdot g(\vec{x}) + z$. To prove (iii) we have $y \in M \;\&\; |x| < |y| \Longleftrightarrow \{\langle 0,1,y \rangle\}^{\mathcal{M}}(x) = 0$; then use (i). The other relations in (iii) are handled similarly. To prove (iv) we consider just the cases of universal and existential quantification. Suppose $Q(\vec{x},y)$ is $\mathcal{M}$-r.e. By (ii) the function

$$f(\vec{x},y) \simeq \begin{cases} 0 & \text{if } Q(\vec{x},y) \text{ is defined} \\ \text{undefined otherwise} \end{cases}$$

is $\mathcal{M}$-partial recursive. Then

$$\begin{aligned} \forall y Q(\vec{x},y) &\Longleftrightarrow \forall y[f(\vec{x},y) \text{ is defined}] \\ &\Longleftrightarrow E(\lambda y f(\vec{x},y)) \text{ is defined.} \end{aligned}$$

To treat existential quantification, let $Q(\vec{x},y) \Longleftrightarrow \{z\}^{\mathcal{M}}(\vec{x},y)$ is defined. Then by A.16 ,

$$\begin{aligned} \exists y Q(\vec{x},y) &\Longleftrightarrow Q(\vec{x}, \nu y Q(\vec{x},y)) \\ &\Longleftrightarrow \{z\}^{\mathcal{M}}(\vec{x}, \nu y Q(\vec{x},y)) \text{ is defined.} \end{aligned}$$

§ A4. Proof of Theorem A5.

A.18. Lemma. There is an $\mathcal{M}$-partial recursive function g such that if $z \in M$ and $\{e\}^{\mathcal{M}}(u,\vec{x})$ is defined for each $u \in M_{|z|}$ then

$$|g(e,z,\vec{x})| = \sup\{|\{e\}^{\mathcal{M}}(u,\vec{x})| : u \in M_{|z|}\} .$$

Proof. Let $g(e,z,\vec{x}) \simeq \nu y Q(e,z,\vec{x},y)$ where

$$Q(e,z,\vec{x},y) \Longleftrightarrow y,z \in M \ \ \& \ \ \forall u \in M_{|z|}[|\{e\}^{\mathcal{M}}(u,\vec{x})| \leq |y|]$$
$$\& \ \forall v \in M_{|y|}\exists u \in M_{|z|}[|v| \leq |\{e\}^{\mathcal{M}}(u,\vec{x})|] .$$

A.19. Lemma. For each ordinal function F primitive recursive in $T_{\mathcal{M}}$ there is an $\mathcal{M}$-partial recursive function h_F such that if $a_1,\ldots,a_n \in M$ then $h_F(a_1,\ldots,a_n) \in M$ and

$$F(|a_1|,\ldots,|a_n|) = |h_F(a_1,\ldots,a_n)| .$$

Proof. We shall use the characterisation of the ordinal functions primitive recursive in $T_{\mathcal{M}}$ given by the following schemata (see [8]):

(i) $F(x,y) = \begin{cases} 0 & \text{if } x \in M_y \\ 1 & \text{otherwise} \end{cases}$

(ii) $F(\vec{x}) = x_i$

(iii) $F(x) = 0$

(iv) $F(x) = x+1$

(v) $F(x,y,u,v) = \begin{cases} x & \text{if } u < v , \\ y & \text{otherwise} . \end{cases}$

(vi) $F(\vec{x},\vec{y}) = G(\vec{x},H(\vec{x}),\vec{y})$

(vii) $F(\vec{x},\vec{y}) = G(H(\vec{x}),\vec{y})$

(viii) $F(z,\vec{x}) = G(\sup_{u<z} F(u,\vec{x}),z,\vec{x})$

For (i) we first need an $\mathcal{M}$-recursive function k such that $|k(n)| = n$ for all n. Let $k(0) = u_0$ and

$$k(n+1) \simeq \nu y[y \in M \ \& \ |k(n)| < |y| \ \& \ \forall z[|z| < |y| \Rightarrow |z| \leq |k(n)|]].$$

Then k has the desired property. Now in case (i) let $|u_1| = 1$ and

$$h_F(a,b) \simeq \begin{cases} u_0 & \text{if } a,b \in M \ \& \ \exists n[|k(n)| = |x| \ \& \ n \in M_{|b|}] \\ u_1 & \text{if } a,b \in M \ \& \ \exists n[|k(n)| = |x| \ \& \ n \notin M_{|b|} \end{cases}$$

Now define h_F in each remaining case as follows:

(ii) $h_F(\vec{a}) = a_i$

(iii) $h_F(a) = u_0$

(iv) $h_F(a) \simeq \nu y[a,y \in M \ \& \ |a| < |y| \ \& \ \forall z[|z| < |y| \Rightarrow |z| \leq |a|]]$

(v) $h_F(a,b,c,d) \simeq \nu y\ [c,d \in M \ \& \ (|c| < |d| \ \& \ y = a) \vee (|d| \leq |c| \ \& \ y=b)]$

(vi) $h_F(\vec{a},\vec{b}) \simeq h_G(\vec{a},h_H(\vec{a}),\vec{b})$

(vii) $h_F(\vec{a},\vec{b}) \simeq h_G(h_H(\vec{a}),\vec{b})$

(viii) $h_F(\vec{a},\vec{b}) \simeq h_G(g(e,\vec{a},\vec{b}),a,\vec{b})$,

where g is from A.18 and e is chosen by the second recursion theorem for $\mathcal{M}$-recursion so that $h_F = \{e\}^{\mathcal{M}}$.

A.20. <u>Lemma</u>. (i) $|M|$ is closed under functions primitive recursive in $T_{\mathcal{M}}$.

(ii) Let $R \subseteq On$ be primitive recursive in $T_{\mathcal{M}}$. Then

$$\{(a_1,\ldots,a_n) : a_1,\ldots,a_n \in M \ \& \ R(|a_1|,|a_2|,\ldots,|a_n|)\}$$

is $\mathcal{M}$-r.e.

Proof. (i) is an immediate consequence of A.19.

Let $A = \{a_1,\ldots,a_n) : a_1,\ldots,a_n \in M \;\&\; R(|a_1|,\ldots,|a_n|)\}$ and F be the representing function of R. Then using A.19,

$$(a_1,\ldots,a_n) \in A \Longleftrightarrow a_1,\ldots,a_n \in M \;\&\; F(|a_1|,\ldots,|a_n|) = 0$$

$$\Longleftrightarrow a_1,\ldots,a_n \in M \;\&\; h_F(a_1,\ldots,a_n) \in M_1 .$$

Thus $A = {}^nM \cap h_F^{-1}M_1$ which is $\mathfrak{M}$-r.e. using A.17.

A.21. Lemma (proof of A.5.1.). $|M| \in \mathrm{Ad}(T_{\mathfrak{M}})$.

Proof. By theorem 3.7 it remains to show that if $R \subseteq {}^3\mathrm{On}$ is primitive recursive in $T_{\mathfrak{M}}$, $\alpha < |M|$ and

(13) $\quad \forall x < |M| \exists y < |M| R(\alpha,x,y)$

then there is $\alpha < \lambda < |M|$ such that

(14) $\quad \forall x < \lambda \exists y < \lambda R(\alpha,x,y)$

Suppose (13) holds. Let $|c| = \alpha$ and $f(x) \simeq \nu y[y \in M \;\&\; R(|c|,|x|,|y|)]$. Then f is $\mathfrak{M}$-partial recursive by A.20. Let g be the $\mathfrak{M}$-recursive function defined by: $g(0) = c$ and

$$g(n+1) = \nu y[y \in M \;\&\; |g(n)| < |y| \;\&\; \forall x \in M_{|g(n)|} |f(x)| \leq |y|] .$$

Then $|g(n)| < |g(n+1)|$ and $|x| < |g(n)| \Rightarrow |f(x)| \leq |g(n+1)|$.
Let $\lambda = \sup\limits_{n<\omega} |g(n)|$. $\alpha < \lambda < |M|$ by A.13 (iv). We show that λ satisfies (14).

(15) $$|x| < \lambda \Longrightarrow \exists n |x| < |g(n)|$$
$$\Longrightarrow \exists n |f(x)| < |g(n)|$$
$$\Longrightarrow |f(x)| < \lambda$$

(14) then follows from (15) and the definition of f.

A.22. <u>Lemma</u> (proof of A.5. (ii)). If $R \subseteq {}^n|M|$ is $|M|$-r.e. in $T_{\mathfrak{M}}$ then there is a primitive recrusive function h such that

$$a_1,\ldots,a_n \in M \ \& \ R(|a_1|,\ldots,|a_n|) \Leftrightarrow h(a_1,\ldots,a_n) \in M .$$

<u>Proof</u>. Let R be $|M|$-r.e. in $T_{\mathfrak{M}}$. Then there is a primitive recursive relation S such that

$$R(\alpha_1,\ldots,\alpha_n) \iff \exists \beta < |M| . S(\beta,\alpha_1,\ldots,\alpha_n) .$$

Let

$$A = \{(a_1,\ldots,a_n) : a_1,\ldots,a_n \in M \ \& \ R(|a_1|,\ldots,|a_n|)\} .$$

Then

$$(a_1,\ldots,a_n) \in A \iff A_1,\ldots,a_n \in M \ \& \exists b[b \in M \ \& \ S(|b|,|a_1|,\ldots,|a_n|)] .$$

It follows that A is $\mathfrak{M}$-r.e. Hence by A.13 there is a recursive function h_1 such that $A = h_1^{-1}({}^nM)$. It remains to find a primitive recursive h. By the S-m-n theorem for ordinary recursive function theory there is a primitive recursive function such that, letting $\vec{a} = a_1,\ldots,a_n$, $\{g(\vec{a})\}(x,M_{|u_o|}) = h_1(\vec{a})$ for all x. Let $h(\vec{a}) = p(u_o,g(\vec{a}))$; then h is primitive recursive and

$$\begin{aligned} (\vec{a}) \in A &\iff h_1(\vec{a}) \in M \\ &\iff \forall x[\{g(\vec{a})\}(x,M_{|u_o|}) \in M \\ &\iff h(\vec{a}) \in M . \end{aligned}$$

<u>Remark</u>. The above proof is the only place where we use the fact that p is <u>primitive</u> recursive instead of just recursive.

A.23. <u>Lemma</u> (proof of A.5. (iii)). Let F be $|M|$-partial recursive in $T_{\mathfrak{M}}$. Then there is a recursive function k such that for $a_1,\dots,a_n \in M$, if $F(|a_1|,\dots,|a_n|)$ is defined then $k(a_1,\dots,a_n) \in M$ and $F(|a_1|,\dots,|a_n|) \leq |k(a_1,\dots,a_n)|$.

Proof. By the normal form theorem relativised to $T_{\mathcal{M}}$, the graph of F is $|M|$-r.e. in $T_{\mathcal{M}}$; hence by A.22. there is a recursive function h such that

$$\vec{a},b \in M \ \&\ F(|a_1|,\dots,|a_n|) = |b| \iff h(a_1,\dots,a_n,b) \in M .$$

Let $f(\vec{a}) \simeq \nu y[y \in M \ \&\ h(a_1,\dots,a_n,y) \in M]$. Then f is $\mathcal{M}$-partial recursive. Hence by A.13. there is a recursive function k such that

$$f(\vec{a}) \in M \implies k(\vec{a}) \in M \ \&\ |f(\vec{a})| \leq |k(\vec{a})| .$$

Then for $\vec{a} \in M$, if $F(|a_1|,\dots,|a_n|)$ is defined

$$F(|a_1|,\dots,|a_n|) = |f(\vec{a})| \leq |k(\vec{a})| .$$

A.24. Lemma (proof of A.5. (iv). $X \subseteq \omega$ is $|M|$-r.e. in $T_{\mathcal{M}}$ iff $X \leq_m M$.

Proof. M is $|M|$-r.e. in $T_{\mathcal{M}}$ since $n \in M \iff \exists\alpha < |M| . T_{\mathcal{M}}(n,\alpha)$. Hence if $X \leq_m M$, X is also $|M|$-r.e. in $T_{\mathcal{M}}$. Now suppose X is $|M|$-r.e. in $T_{\mathcal{M}}$, and let k be an $\mathcal{M}$-recursive function such that $|k(n)| = n$ for all n (see the proof of A.19.). Then using A.22. there is a recursive function h such that

$$\begin{aligned} n \in X &\iff |k(n)| \in X \\ &\iff hk(n) \in M . \end{aligned}$$

X is the inverse image of the $\mathcal{M}$-r.e. set M under the $\mathcal{M}$-recursive function hk and hence is $\mathcal{M}$-r.e. by A.17. (iv). Hence $X \leq_m M$ by A.13. (i).

Remarks. (i) The definition of an acceptable ordinal system differs slightly from that given in [15]. The requirement that j (called there g) be a many-one reduction of M to M is necessary for the proof of A.12 and its omission was an oversight in [15]. The other change is the requirement that p be primitive recursive instead of recursive, and as mentioned above this is only to ensure that the function h of A.5 is primitive recursive.

(ii) A.5. (iii) is not used elsewhere but it appears to be of interest in its own right and its proof comes naturally from our construction. It was used in [15] in an earlier proof of some of our results but is not needed in our present formulation.

(iii) The method we have used in proving the Coding Lemma is to utilize techniques from the well-developed theory of recursive functionals of type 2. In particular, the crucial results needed about $\mathcal{M}$-recursion, namely the boundedness theorem (A.13. (iv)), and theorem A.15 on the existence of selection functions are proved by standard methods from the theory of recursive functionals of type 2. On the other hand the theory of recursive functionals may be regarded as a part of the theory of inductive definitions. This suggests that an ultimately simpler and more elegant proof of the Coding Lemma in a more general setting can be provided within the "pure" theory of inductive definitions.

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